SOME INEQUALITIES IN THE CONVEX QUADRILATERAL

NICUȘOR MINCULETE¹, MIHÁLY BENČZE², OVIDIU T. POP³

¹Dimitrie Cantemir University, 500068, Brașov, Romania
²Aprily Lájos National College, 500026, Brașov, Romania
³Mihai Eminescu National College, 440014, Satu Mare, Romania

Abstract: In this paper we present some inequalities about the convex quadrilateral using the Jensen Inequality, the Tóth-Lenhard Inequality.

Keywords: convex quadrilateral, Jensen Inequality, Tóth-Lenhard Inequality

1. INTRODUCTION

Let \( A_1A_2A_3A_4 \) be a convex quadrilateral and \( M \) an interior point. Florian [1-3] proved the following inequality for a convex quadrilateral:

\[
\sqrt{2}(w_1 + w_2 + w_3 + w_4) \leq MA_1 + MA_2 + MA_3 + MA_4,
\]

(1)

where \( w_1, w_2, w_3 \) and \( w_4 \) are the bisectors of the angles \( A_1MA_2, A_2MA_3, A_3MA_4 \) and \( A_4MA_1 \) respectively.

But \( p_1 \leq w_1, p_2 \leq w_2, p_3 \leq w_3, p_4 \leq w_4 \), where \( p_1, p_2, p_3, p_4 \), are the distances from \( M \) to the sides of the convex quadrilateral \( A_1A_2A_3A_4 \) (\( p_i \) is the distance from \( M \) to \( A_iA_{i+1} \)) etc, hence:

\[
\sqrt{2}(p_1 + p_2 + p_3 + p_4) \leq MA_1 + MA_2 + MA_3 + MA_4.
\]

(2)

Other inequality of Erdös-Mordell-type, for the convex quadrilateral, was given by N. Ozeki [4] in 1957, namely,

\[
4 \cdot w_1 \cdot w_2 \cdot w_3 \cdot w_4 \leq MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4,
\]

(3)

which proved the inequality:

\[
4 \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \leq MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4.
\]

(4)

Denote by \( r_i, r_2, r_3, r_4 \) the radii of circles (circumscribing the triangles) \( MA_1A_2, MA_2A_3, MA_3A_4, MA_4A_1 \) respectively. Next, we establish some inequalities similar of about inequalities, between the lengths \( MA_1, MA_2, MA_3, MA_4 \) and the radii \( r_1, r_2, r_3, r_4 \).

2. MAIN RESULTS

**Theorem 2.1.** For every convex quadrilateral \( A_1A_2A_3A_4 \) the following inequality

\[
MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 \leq 4r_1r_2r_3r_4,
\]

(5)

holds.

**Proof:** By applying the sine rule, we deduce the relations (see figure 1)

\[
MA_1 = 2r_1 \sin A_{1,1} \sin A_{1,2},
\]

\[
MA_2 = 2r_2 \sin A_{2,1} \sin A_{2,2},
\]

hence:

\[
\frac{MA_1^2}{4r_1r_2} = \sin A_{1,1} \sin A_{1,2}
\]

Similarly, we obtain the equalities:
\[
\frac{MA_3^2}{4R_3R_3} = \sin A_{2,1} \sin A_{4,2}, \quad \frac{MA_4^2}{4R_4R_4} = \sin A_{3,1} \sin A_{1,2} \quad \text{and} \quad \frac{MA_2^2}{4R_2R_2} = \sin A_{4,1} \sin A_{2,2}.
\]

Taking the product of the four equalities, we obtain
\[
\left(\frac{MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4}{16R_1R_2R_3R_4}\right)^2 = \sin A_{2,1} \sin A_{4,2} \sin A_{3,1} \sin A_{1,2} \sin A_{3,1} \sin A_{2,2} \sin A_{4,1} \sin A_{4,2},
\]

But when applying the Jensen inequality for the concave function, \(\ln (\sin x)\), it follows that
\[
\sin A_{2,1} \sin A_{4,2} \sin A_{3,1} \sin A_{2,2} \sin A_{4,1} \sin A_{4,2} \leq \frac{1}{16},
\]
so:
\[
\frac{MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4}{16R_1R_2R_3R_4} \leq \frac{1}{4},
\]
consequently:
\[
MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 \leq 4R_1R_2R_3R_4.
\]

**Theorem 2.2.** In any convex quadrilateral \(A_1, A_2, A_3, A_4\) there is the inequality
\[
MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 \leq 16R_1R_2R_3R_4 \sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} \sin \frac{A_4}{2}. \tag{6}
\]

**Proof.** We have the relation
\[
\frac{MA_1}{2R_1} + \frac{MA_2}{2R_2} = \sin A_{1,1} + \sin A_{1,2} \leq 2 \sin \frac{A_{1,1} + A_{1,2}}{2} = 2 \sin \frac{A_1}{2}, \tag{7}
\]
but:
\[
\sqrt{\frac{MA_1 \cdot MA_2}{R_1 \cdot R_2}} \leq 2 \sin \frac{A_1}{2}, \tag{8}
\]
which means that:
\[
\frac{MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4}{16R_1R_2R_3R_4} \leq \sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} \sin \frac{A_4}{2},
\]
so:
\[
MA_1 \cdot MA_2 \cdot MA_3 \cdot MA_4 \leq 16R_1R_2R_3R_4 \sin \frac{A_1}{2} \sin \frac{A_2}{2} \sin \frac{A_3}{2} \sin \frac{A_4}{2}.
\]

**Theorem 2.3.** If \(x_1, x_2, x_3, x_4 \in \mathbb{R}^+\), then in any convex quadrilateral \(A_1, A_2, A_3, A_4\) there are the inequalities
\[
\left(\frac{x_1 x_2}{\sqrt{R_1 R_2}} + \frac{x_3 x_4}{\sqrt{R_3 R_4}}\right)\sqrt{MA_2 \cdot MA_4} + \left(\frac{x_3 x_3}{\sqrt{R_2 R_1}} + \frac{x_4 x_4}{\sqrt{R_4 R_3}}\right)\sqrt{MA_3 \cdot MA_1} \leq \frac{1}{2} \left[\left(\frac{x_2 x_3}{R_1} + \frac{x_4 x_4}{R_4}\right)MA_1 + \left(\frac{x_1 x_2}{R_2} + \frac{x_3 x_4}{R_2}\right)MA_2 + \left(\frac{x_2 x_3}{R_2} + \frac{x_4 x_4}{R_3}\right)MA_3 + \left(\frac{x_1 x_2}{R_4} + \frac{x_3 x_4}{R_4}\right)MA_4\right] \leq \sqrt{2} \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right). \tag{9}
\]

**Proof.** We know from [1-3] the Tóth- Lenhard Inequality:

*Let \(x_1, x_2, \ldots, x_n\) and \(\theta_1, \theta_2, \ldots, \theta_n\) be two sets of positive real numbers so that \(\theta_1 + \theta_2 + \ldots + \theta_n = \pi\).*

If we put \(x_{n+1} = x_1\), then:
\[
\sum_{k=1}^{n} x_k x_{k+1} \cos \theta_k \leq \cos \left(\frac{\pi}{n}\right) \sum_{k=1}^{n} x_k^2.
\]

We choose \(n = 4\) and \(\theta_k = \frac{\pi - A_k}{2}, \quad (\forall) k = 1, 4\). We remark that \(\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi\).
Hence we applying the Tóth- Lenhard Inequality, we have:
\[
\sum_{k=1}^{4} x_k x_{k+1} \sin \frac{A_k}{2} = \sum_{k=1}^{4} x_k x_{k+1} \cos \theta_k \leq \cos \left( \frac{\pi}{4} \right) \sum_{k=1}^{4} x_k^2,
\]
so:
\[
\sum_{k=1}^{4} x_k x_{k+1} \sin \frac{A_k}{2} \leq \frac{\sqrt{2}}{2} \sum_{k=1}^{4} x_k^2.
\] (10)

We multiply the relations (7) and (8) with \( \frac{12}{x_k} \), hence
\[
2 \sin^2 2 \sqrt{A_k} \leq \left( \frac{MA_2 + MA_4}{2R_1} \right) x_1 x_2 \leq 2x_1 x_2 \sin \frac{A_1}{2}.
\]
We written the similarly relation and taking the sum, we obtain the sequence of inequalities
\[
\left( \frac{x_1 x_2 + x_3 x_4}{\sqrt{R_1 R_4}} + \frac{x_3 x_4}{\sqrt{R_2 R_3}} \right) \sqrt{MA_2 \cdot MA_4} + \left( \frac{x_2 x_3}{\sqrt{R_2 R_1}} + \frac{x_4 x_1}{\sqrt{R_3 R_4}} \right) \sqrt{MA_3 \cdot MA_1} \leq \frac{1}{2} \left( \frac{x_1 x_2 + x_3 x_4}{R_1} \right) MA_1 + \frac{x_1 x_2}{R_2} \frac{x_3 x_4}{R_4} \frac{x_3 x_4}{R_2} \frac{x_4 x_1}{R_3} \frac{x_4 x_1}{R_3}
\]
\[
\leq 2x_1 x_2 \sin \frac{A_1}{2} + 2x_2 x_3 \sin \frac{A_2}{2} + 2x_3 x_4 \sin \frac{A_3}{2} + 2x_4 x_1 \sin \frac{A_4}{2} \leq \sqrt{2} \left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right).
\]

Therefore, we obtain the inequalities of statement.

**Corollary 2.4.** In any convex quadrilateral \( A_1 A_2 A_3 A_4 \), the inequalities
\[
\left( \frac{1}{\sqrt{R_1 R_4}} + \frac{1}{\sqrt{R_2 R_3}} \right) \sqrt{MA_2 \cdot MA_4} + \left( \frac{1}{\sqrt{R_2 R_1}} + \frac{1}{\sqrt{R_3 R_4}} \right) \sqrt{MA_3 \cdot MA_1} \leq \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_4} \right) MA_1 + \frac{1}{R_2} + \frac{1}{R_3} \left( \frac{1}{R_2} + \frac{1}{R_3} \right) MA_2 + \left( \frac{1}{R_2} + \frac{1}{R_3} \right) MA_3 + \left( \frac{1}{R_2} + \frac{1}{R_3} \right) MA_4 \leq 4\sqrt{2}
\] (11)

and
\[
\left( \frac{R_2}{R_4} + \frac{R_4}{R_2} \right) \sqrt{MA_2 \cdot MA_4} + \left( \frac{R_1}{R_3} + \frac{R_3}{R_1} \right) \sqrt{MA_1 \cdot MA_3} \leq \frac{1}{2} \left( \frac{\sqrt{R_2 R_3}}{R_1} + \frac{\sqrt{R_1 R_4}}{R_4} \right) MA_1 + \left( \frac{\sqrt{R_3 R_4}}{R_2} + \frac{\sqrt{R_2 R_1}}{R_1} \right) MA_2 + \left( \frac{\sqrt{R_4 R_1}}{R_3} + \frac{\sqrt{R_3 R_2}}{R_2} \right) MA_3 + \left( \frac{\sqrt{R_4 R_1}}{R_3} + \frac{\sqrt{R_3 R_2}}{R_2} \right) MA_4 \leq \sqrt{2}(R_1 + R_2 + R_3 + R_4)
\] (12)

hold.

**Proof.** It follows from inequality (9) if we put \( x_1 = x_2 = x_3 = x_4 = 1 \) and \( x_1 = \sqrt{R_1} \), \( x_2 = \sqrt{R_2} \), \( x_3 = \sqrt{R_3} \), \( x_4 = \sqrt{R_4} \) respectively.

**Corollary 2.5.** In any convex quadrilateral \( A_1 A_2 A_3 A_4 \) there are the inequalities
\[
\frac{MA_1}{\sqrt{R_4 R_1}} + \frac{MA_2}{\sqrt{R_1 R_2}} + \frac{MA_3}{\sqrt{R_2 R_3}} + \frac{MA_4}{\sqrt{R_3 R_4}} \leq 4\sqrt{2}
\] (13)
and \[ \sqrt{2\left(\sqrt{MA_1 \cdot MA_3} + \sqrt{MA_2 \cdot MA_4}\right)} \leq R_1 + R_2 + R_3 + R_4. \] (14)

**Proof.** From the relation (11), we have
\[ \left(\frac{1}{R_1} + \frac{1}{R_4}\right)MA_1 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)MA_3 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)MA_4 + \left(\frac{1}{R_3} + \frac{1}{R_4}\right)MA_2 \leq 8\sqrt{2} \]
and the relation \[ \frac{1}{u} + \frac{1}{v} \geq \frac{2}{\sqrt{uv}}, \] for all \( u, v > 0 \), implies that
\[ \frac{MA_1}{\sqrt{R_1 R_4}} + \frac{MA_2}{\sqrt{R_1 R_3}} + \frac{MA_3}{\sqrt{R_2 R_4}} + \frac{MA_4}{\sqrt{R_2 R_3}} \leq 4\sqrt{2}. \]
From the relation (11), we have
\[ \left(\frac{R_2}{R_4} + \frac{R_4}{R_2}\right)\sqrt{MA_1 \cdot MA_3} + \left(\frac{R_1}{R_3} + \frac{R_3}{R_1}\right)\sqrt{MA_2 \cdot MA_4} \leq \sqrt{2}(R_1 + R_2 + R_3 + R_4) . \]
But, using the inequality \[ \sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}} \geq 2, \] for all \( u, v > 0 \), we deduce inequality (14).

**Remark.** From inequality (13), we deduce another inequality
\[ \frac{MA_1}{R_1 + R_4} + \frac{MA_2}{R_1 + R_2} + \frac{MA_3}{R_2 + R_3} + \frac{MA_4}{R_3 + R_4} \leq 2\sqrt{2} . \] (15)

**Corollary 2.6.** In any convex quadrilateral \( A_1A_2A_3A_4 \), the inequalities
\[ \sqrt{2\left(\sqrt{MA_1 \cdot MA_3} + \sqrt{MA_2 \cdot MA_4}\right)} \leq \sqrt{R_1 R_2 R_3 R_4 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)} \] (16)
and
\[ (R_2 + R_3)MA_1 + (R_3 + R_4)MA_2 + (R_4 + R_1)MA_3 + (R_1 + R_2)MA_4 \leq 2\sqrt{2R_1 R_2 R_3 R_4 \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} + \frac{1}{R_4^2}\right)} \]
hold.

**Proof.** It easy to see that from inequality (9) that for \( x_1 = \frac{1}{\sqrt{R_2}}, \ x_2 = \frac{1}{\sqrt{R_3}} \),
\[ x_3 = \frac{1}{\sqrt{R_4}}, \ x_4 = \frac{1}{\sqrt{R_1}} \]
and \( x_1 = \frac{1}{R_2}, \ x_2 = \frac{1}{R_3}, \ x_3 = \frac{1}{R_4}, \ x_4 = \frac{1}{R_1} \) respectively, we obtain inequalities (16) and (17).

**REFERENCES**


Manuscript received: 18.03.2010
Accepted paper: 28.05.2010
Published online: 22.06.2010