A GENERALIZATION OF BERGSTROM AND RADON’S INEQUALITIES IN PSEUDO-HILBERT SPACES

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Abstract. In this note we had presented two generalizations for Bergstrom and Radon’s inequalities for seminorms in pseudo-Hilbert spaces and in normed spaces. Some applications are given, as well.

Keywords: pseudo-Hilbert spaces (Loynes spaces), seminorms, Bergstrom inequality, Radon inequality.

INTRODUCTION

First we need to recall, see [3, 5], that a locally convex space \( Z \) is called admissible in the Loynes sense if the following conditions are satisfied:

- \( Z \) is complete;
- there is a closed convex cone in \( Z \), denoted \( Z_+ \), that defines an order relation on \( Z \) (that is \( z_1 \leq z_2 \) if \( z_1 - z_2 \in Z_+ \));
- there is an involution in \( Z \), \( Z \ni z \mapsto z^* \in Z \) (that is \( z^{**} = z \), \((\alpha z)^* = \overline{\alpha} z^*\), \((z_1 + z_2)^* = z_1^* + z_2^*\)), such that \( z \in Z_+ \) implies \( z^* = z \);
- the topology of \( Z \) is compatible with the order (that is, there exists a basis of convex solid neighbourhoods of the origin);
- and any monotonously decreasing sequence in \( Z_+ \) is convergent.

We shall say that a set \( C \in Z \) is called solid if \( 0 \leq z' \leq z'' \) and \( z'' \in C \) implies \( z' \in C \).

As an easy example we shall consider, \( Z = C \), a \( C^* \) - algebra with topology and natural involution.

Let \( Z \) be an admissible space in the Loynes sense. A linear topological space \( \mathcal{H} \) is called pre-Loynes \( Z \)-space if it satisfies the following properties:

- \( \mathcal{H} \) is endowed with a \( Z \) - valued inner product (gramian), i.e. there exists an application
  \[ \mathcal{H} \times \mathcal{H} \ni (h, k) \mapsto [h, k] \in Z \]
  having the properties: \( [h, h] \geq 0 \), \( [h, h]=0 \) implies \( h=0 \);
  \( [h_1+h_2, h]=[h_1, h]+[h_2, h] \); \( [\lambda h, k]=\lambda [h, k] \); \( [h, k]^*=[k, h] \);
for all \( h, k, h_1, h_2 \in \mathcal{H} \) and \( \lambda \in C \).
- The topology of \( \mathcal{H} \) is the weakest locally convex topology on \( \mathcal{H} \) for which the application \( \mathcal{H} \ni h \mapsto [h, h] \in Z \) is continuous. Moreover, if \( \mathcal{H} \) is a complete space with this topology, then \( \mathcal{H} \) is called Loynes \( Z \)-space.

Now, considering \( Z=C \) as above, \( Z \) with \( [z_1, z_2] = z_2^* z_1 \) is a Loynes \( Z \)-space.

An important result which can be used below is given in the next statement, and was proved in [5].

Let \( \mathcal{H} \) and \( \mathcal{K} \) be two Loynes \( Z \)-spaces.

We recall that in [3-5] an operator \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is called gramian bounded, if there exists a constant \( \mu>0 \) such that in the sense of order of \( Z \) holds

\[ [Th, Th]_K \leq \mu [h, h]_H, \ h \in \mathcal{H}. \]  \( (1) \)
We denote the class of such operators by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and

$\mathcal{B}'(\mathcal{H}, \mathcal{K}) = \mathcal{B}(\mathcal{H}, \mathcal{K}) \cap L^*(\mathcal{H}, \mathcal{K})$.

We also denote the introduced norm by

$$||T|| = \inf\{\sqrt{\mu} \mid \mu > 0 \text{ and satisfies } (1)\}.$$  \hfill (2)

It is known that the space $\mathcal{B}'(\mathcal{H}, \mathcal{K})$ is a Banach space, and its involution $\mathcal{B}'(\mathcal{K}, \mathcal{H})$ in $\mathcal{B}'(\mathcal{K}, \mathcal{H})$ satisfies

$$||T^*|| = ||T||^2, T \in \mathcal{B}'(\mathcal{H}, \mathcal{K}).$$

In particular $\mathcal{B}'(\mathcal{H})$ is a $C^*$--algebra.

The following two results were presented in [3].

**Lemma 1.** If $p$ is a continuous and monotonous seminorm on $\mathcal{Z}$, then:

$$q_p(h) = (p([h, h]))^{\frac{1}{2}}$$

is a continuous seminorm on $\mathcal{H}$.

**Proposition 1.** If $\mathcal{H}$ is a pre-Loynes $\mathcal{Z}$-space and $\mathcal{P}$ is a set of monotonous (increasing) seminorms defining the topology of $\mathcal{Z}$, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_p = \{q_p \mid p \in \mathcal{P}\}$.

We suppose that $mq_{p_2}(x) \leq q_{p_1}(x) \leq Mq_{p_2}(x), (\forall) x \in \mathcal{H}$, with $P_1, P_2$ continuous and increasing seminorms on $\mathcal{Z}$ and $M$ finite, $M \geq m > 0$. Because $P_2$ is increasing, we have:

$$\frac{M^2}{m^2} \left\{ 2 + \frac{p_1([x, y] + [y, x])}{q_{p_1}(x)q_{p_1}(y)} \right\} \geq q_{p_2} \left( x + \frac{y}{q_{p_2}(y)} \right) \geq q_{p_2} \left( \frac{q_{p_1}(x)q_{p_2}(y)}{q_{p_2}(x)q_{p_2}(y)} \right)$$

Thus, for example,

$$\frac{p_2([x, y] + [y, x])}{q_{p_2}(x)q_{p_2}(y)} \leq \frac{M^2}{m^2} \left\{ 1 + \frac{1}{2} \frac{p_1([x, y] + [y, x])}{q_{p_1}(x)q_{p_1}(y)} \right\}.$$

Let $\mathcal{Z}$ be an admissible space in the Loynes sense and $\mathcal{H}$ is a pre-Loynes $\mathcal{Z}$-space. Using the Radon's inequality we can state:

**Remark 1.** If $h_k \in \mathcal{H}$, $a_k > 0$ with $q_p(h_k) > 0$, $r > 0$, $k \in (1, 2, ..., n)$ then we shall have:

$$\sum_{k=1}^{n} \frac{q_p(h_k)^r}{a_k^{r+1}} \geq \left( \frac{\sum_{k=1}^{n} q_p(h_k)^{r+1}}{\sum_{k=1}^{n} a_k^r} \right)^r$$

When we take $r = 1$ a variant of Bergstrom's inequality with seminorm is obtained.

2. THE MAIN RESULTS

The proofs of the Theorems 1 and 4 will use the same techniques as in [6].

**Theorem 1.** For $a_k, h_k \in \mathcal{H}$ with $q_p(h_k) > 0$, $r \geq 1$, $k \in (1, 2, ..., n)$, $n \geq 2$, $n \in \mathbb{N}$ the following inequality takes place:
\[
\sum_{i=1}^{n} \frac{q_p(h_i)^{-q}}{a_k^{r}} \geq \sum_{i=1}^{n} \frac{q_p(h_i)^{-q}}{a_k^{r}} \geq \max_{1 \leq i < j \leq n} \frac{q_p(h_i)^{-q} + q_p(h_j)^{-q} - q_p(h_i + h_j)^{-q}}{(a_i + a_j)^r}
\]

**Proof:** We shall consider the sequence,

\[
d_n = \frac{q_p(h_1)^{-q}}{a_1^{r}} + \ldots + \frac{q_p(h_n)^{-q}}{a_n^{r}} - \frac{q_p(h_1 + \ldots + h_n)^{-q}}{(a_1 + \ldots + a_n)^r}
\]

and we shall prove that \((d_n)n\) is an increasing monotonous sequence.

This fact it indeed true if consider

\[
d_{n+1} - d_n = \frac{q_p(h_{n+1})^{-q}}{a_{n+1}^{r}} - \frac{q_p(h_1 + \ldots + h_n + h_{n+1})^{-q}}{(a_1 + \ldots + a_n + a_{n+1})^{r}} + \frac{q_p(h_1 + \ldots + h_n)^{-q}}{(a_1 + \ldots + a_n)^r}
\]

because \(q_p\) is seminorm which implies,

\[
q_p(h_1 + \ldots + h_n + h_{n+1}) \leq q_p(h_1 + \ldots + h_n) + q_p(h_{n+1})
\]

and also:

\[
\frac{(q_p(h_1 + \ldots + h_n + h_{n+1}))^{-q}}{((a_1 + \ldots + a_n) + a_{n+1})^{r}} \leq \frac{q_p(h_1 + \ldots + h_n) + q_p(h_{n+1})^{-q}}{((a_1 + \ldots + a_n) + a_{n+1})^{r}} \leq \frac{q_p(h_1 + \ldots + h_n)^{-q}}{(a_1 + \ldots + a_n)^r} + \frac{q_p(h_{n+1})^{-q}}{a_{n+1}^{r}}.
\]

We used before the Radon's inequality applied for \(n=2\), see [1, 6],

\[
\alpha^{-q} + \beta^{-q} \geq (\alpha + \beta)^{-q}, \quad (3)
\]

and we took \(\alpha = q_p(h_1 + \ldots + h_n)\), \(\beta = q_p(h_{n+1})\), \(a = a_1 + \ldots + a_n\) and \(b = a_{n+1}\)

Another proof for inequality (3), can be found in [1].

The sequence \((d_n)n\) being increasing, we obtain that,

\[
d_n \geq d_{n-1} \geq \ldots \geq d_2 \geq d_1 = 0
\]

and that also means that

\[
d_n \geq d_2 = \frac{q_p(h_1)^{-q}}{a_1^{r}} + \frac{q_p(h_2)^{-q}}{a_2^{r}} - \frac{q_p(h_1 + h_2)^{-q}}{(a_1 + a_2)^r}, \quad (\forall)n \in \mathbb{N}, \quad n \geq 2.
\]

The symmetry of \(d_n\) relatively to the variables \(a_i\) and \(h_j\), \(i, j \in \{1, 2, \ldots, n\}\) allows us to notice that

\[
d_n \geq \frac{q_p(h_i)^{-q}}{a_i^{r}} + \frac{q_p(h_j)^{-q}}{a_j^{r}} - \frac{q_p(h_i + h_j)^{-q}}{(a_i + a_j)^r}, \quad (\forall)n \in \mathbb{N}, \quad n \geq 2, i, j \in \{1, 2, \ldots, n\}.
\]

For \(r=1\) is obtained below a refinement of Bergstrom's inequality.

**Corollary 1.** For \(a_k > 0, h_k \in H, k \in \{1, 2, \ldots, n\}\), \(n \geq 2, n \in \mathbb{N}\) we shall obtain the following inequality:
A generalization of Bergstrom and Radon's …

Theorem 2. For \( a_k > 0, x_k \in X, r \geq 0, k \in \{1, 2, ..., n\}, n \geq 2, n \in \mathbb{N} \) and every arbitrary seminorm \( p: X \rightarrow \mathbb{R}^+ \) we have:

\[
\sum_{k=1}^{n} \left( \frac{\sum_{i=1}^{n} p(x_i)^{r^*}}{a_k} \right) \geq \max_{1 \leq i < j \leq n} \left( \frac{p(x_i)^{r^*} + p(x_j)^{r^*}}{a_i^{r^*} + a_j^{r^*}} - \frac{p(x_i + x_j)^{r^*}}{(a_i + a_j)^{r^*}} \right).
\]

Corollary 2. In fact with the above conditions, the Corollary 1 remains true for every seminorm \( p \) from a family of seminorms which defines the topology of the linear space considered instead of \( q_p \).

Theorem 3. If we consider a normed space \( \mathcal{H}, x_k \in \mathcal{H}, k \in \{1, 2, ..., n\} \) and with the above conditions of the Theorem 1, then we have the following inequality:

\[
\sum_{k=1}^{n} \left( \frac{\sum_{i=1}^{n} \| x_i \|^{r^*}}{a_k^{r^*}} \right) \geq \max_{1 \leq i < j \leq n} \left( \frac{\| x_i \|^{r^*} + \| x_j \|^{r^*}}{a_i^{r^*} + a_j^{r^*}} - \frac{\| x_i + x_j \|^{r^*}}{(a_i + a_j)^{r^*}} \right).
\]

Proof:

It will be as the proof of the Theorem 1, we shall only take

\[
d_n = \frac{\| x_1 \|^{r^*} + \cdots + \| x_n \|^{r^*}}{a_1^{r^*} + \cdots + a_n^{r^*}} - \frac{\| x_1 + \cdots + x_n \|^{r^*}}{(a_1 + \cdots + a_n)^{r^*}}
\]

and \( \alpha = \| x_1 + \cdots + x_n \|, \ \beta = \| x + e_q \|, \ a = a_1 + \cdots + a_n \) and \( b = a_{n+1} \) in relation (3).

Remark 2. We can consider instead of seminorm \( p \), a norm \( \| \| \), in a normed space \( \mathcal{H}, x_k \in \mathcal{H} \) and then under conditions of the above corollary we shall have,

\[
\sum_{k=1}^{n} \left( \frac{\sum_{i=1}^{n} \| x_i \|^2}{a_k} \right) \geq \max_{1 \leq i < j \leq n} \left( \frac{(a_i + a_j)(\| x_i \|^2 + \| x_j \|^2)}{a_i^{r^*} + a_j^{r^*}} - \frac{(a_i + a_j)\| x_i + x_j \|^2}{(a_i + a_j)^{r^*}} \right).
\]

In what follows we shall present a generalizations of the Remark 1 concerning the Radon's inequality for seminorms \( q_p \).

Remark 3. If \( h_k \in \mathcal{H}, a_k > 0, r > 0, s \geq 1, k \in \{1, 2, ..., n\}, m \geq 1, \) then the following inequalities take place:

\[
\sum_{k=1}^{n} \left( \frac{\sum_{i=1}^{n} q_p(h_i)^{r^*}}{a_k} \right) \geq \max_{1 \leq i < j \leq n} \left( \frac{\sum_{i=1}^{n} q_p(h_i)^{r^*} + \sum_{i=1}^{n} q_p(h_i)^{r^*}}{a_i^{r^*} + a_j^{r^*}} - \frac{\sum_{i=1}^{n} q_p(h_i)^{r^*}}{(a_i + a_j)^{r^*}} \right).
\]

Now we shall be able to give a generalization of Theorem 1, Radon's inequality for seminorms \( q_p \).

Theorem 4. For \( a_k, h_k \in \mathcal{H} \) with \( q_p(h_k) > 0, r \geq 0, s \geq 1, k \in \{1, 2, ..., n\}, n \geq 2, n \in \mathbb{N} \) the following inequality takes place:

\[
\sum_{k=1}^{n} \left( \frac{\sum_{i=1}^{n} q_p(h_i)^{r^*}}{a_k} \right) \geq \max_{1 \leq i < j \leq n} \left( \frac{\sum_{i=1}^{n} q_p(h_i)^{r^*} + \sum_{i=1}^{n} q_p(h_i)^{r^*}}{a_i^{r^*} + a_j^{r^*}} - \frac{\sum_{i=1}^{n} q_p(h_i)^{r^*}}{(a_i + a_j)^{r^*}} \right).
\]
A generalization of Bergstrom and Radon’s inequality

\[
\sum_{i=1}^{n} q_{p}(h_{i}) \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{q_{p}(h_{i})}{a_{i}^{r}} + \frac{q_{p}(h_{j})}{a_{j}^{r}} - \frac{q_{p}(h_{i} + h_{j})}{(a_{i} + a_{j})^{r}} \right). \quad (4)
\]

**Proof:** We shall write:
\[
\sum_{i=1}^{n} q_{p}(h_{i}) \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{q_{p}(h_{i})}{a_{i}^{r}} + \frac{q_{p}(h_{j})}{a_{j}^{r}} - \frac{q_{p}(h_{i} + h_{j})}{(a_{i} + a_{j})^{r}} \right).
\]

and then applying the inequality from Theorem 1, we shall obtain,
\[
\sum_{i=1}^{n} q_{p}(h_{i}) \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{q_{p}(h_{i})}{a_{i}^{r}} + \frac{q_{p}(h_{j})}{a_{j}^{r}} - \frac{q_{p}(h_{i} + h_{j})}{(a_{i} + a_{j})^{r}} \right).
\]

This inequality becomes,
\[
\sum_{i=1}^{n} q_{p}(h_{i}) \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{q_{p}(h_{i})}{a_{i}^{r}} + \frac{q_{p}(h_{j})}{a_{j}^{r}} - \frac{q_{p}(h_{i} + h_{j})}{(a_{i} + a_{j})^{r}} \right).
\]

Now, using the inequality from Remark 3, we shall obtain:
\[
\sum_{i=1}^{n} q_{p}(h_{i}) \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{q_{p}(h_{i})}{a_{i}^{r}} + \frac{q_{p}(h_{j})}{a_{j}^{r}} - \frac{q_{p}(h_{i} + h_{j})}{(a_{i} + a_{j})^{r}} \right).
\]

**Remark 4.** In fact under the above conditions, the above theorem remains true for every seminorm \( p \) from a family of seminorms which defines the topology of the linear space considered instead of \( q_{p} \):
\[
\sum_{i=1}^{n} p(x_{i}) \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{p(x_{i})}{a_{i}^{r}} + \frac{p(x_{j})}{a_{j}^{r}} - \frac{p(x_{i} + x_{j})}{(a_{i} + a_{j})^{r}} \right), \quad (5)
\]

(\( \forall \) \( x_{i} \in X \), with \( p(x_{i}) > 0 \).
- Moreover, in every normed space \( X \), we have under above conditions,
\[
\sum_{i=1}^{n} \| x_{i} \| \geq \frac{1}{n^{-1}} \left( \sum_{i=1}^{n} a_{i}^{r} \right)^{\gamma} + \max_{1 \leq i < j \leq n} \left( \frac{\| x_{i} \|}{a_{i}^{r}} + \frac{\| x_{j} \|}{a_{j}^{r}} - \frac{\| x_{i} + x_{j} \|}{(a_{i} + a_{j})^{r}} \right), \quad (6)
\]

(\( \forall \) \( x_{i} \in X \).
- Finally, for \( a_{k}, h_{k} \in H \) with \( q_{p}(h_{k}) > 0 \), \( r \geq 0 \), \( s \geq r+1 \), \( k \in \{1, 2, \ldots, n\} \), \( n \geq 2 \), \( n \in \mathbb{N} \) a variant of Radon’s inequality takes place:
\[ \sum_{k=1}^{n} \frac{q_p(h_k)^r}{a_k^r} = \frac{1}{n^{i-r-1}} \left( \sum_{i=1}^{n} q_p(h_i)^r \right)^{s} + \max_{1 \leq \iota \leq \eta \leq n} \left( \frac{q_p(h_\iota)^r}{a_\iota^r} + \frac{q_p(h_\eta)^r}{a_\eta^r} - \frac{q_p(h_\iota + h_\eta)^r}{(a_\iota + a_\eta)^r} \right). \] (7)

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Manuscript received: 16.08.2009
Accepted paper: 30.04.2010
Published online: 04.10.2010