ON A CONVERGENCE BY DETEMPLE

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Abstract. The aim of this paper is to discuss the sequence defined by DeTemple in [6].

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INTRODUCTION

The Euler-Mascheroni constant \( \gamma = 0.577215664 \ldots \) is defined as the limit of:

\[
D_n = H_n - \ln n
\]  

(1.1)

where \( H_n \) denotes the \( n \)th harmonic number, defined for \( n \in \mathbb{N} \) by \( H_n = \sum_{k=1}^{n} \frac{1}{k} \).

Several bounds for \( D_n - \gamma \) have been given in the literature [2, 3, 5, 7-9]. The convergence of the sequence \( D_n \) to \( \gamma \) is very slow. Some quicker approximations to the Euler-Mascheroni constant were established and we mention here the following sequence introduced by DeTemple [6]:

\[
R_n = H_n - \ln \left( n + \frac{1}{2} \right),
\]

for which

\[
\frac{1}{24(n + 1)^2} < R_n - \gamma < \frac{1}{24n^2}
\]  

(1.2)

First we use the asymptotic series of the digamma function \( \psi \) in terms of Bernoulli numbers

\[
\psi(x + 1) = \ln x + \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} x^{2k} = \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots
\]

(1.3)

(see, e.g., [1]) to deduce the standard asymptotic series of DeTemple's sequence

\[
\sum_{k=1}^{n} \frac{1}{k} - \ln \left( n + \frac{1}{2} \right) \sim \gamma + \frac{1}{24n^2} - \frac{1}{24n^3} + \frac{23}{960n^4} - \frac{1}{160n^5} - \frac{11}{8064n^6} + \ldots
\]

(1.4)

Recently, Chen [4] obtained the following sharp form of the inequality (1.2):

\[
\frac{1}{24(n + a)^2} \leq R_n - \gamma < \frac{1}{24(n + b)^2}, \quad n \geq 1
\]

(1.5)

with the best possible constants

\[
a = \frac{1}{\sqrt{24 \left( -\gamma + 1 - \ln \frac{3}{2} \right)}} = -1.055106\ldots \quad \text{and} \quad b = \frac{1}{2}
\]

We propose the following series in negative powers of \( n - 1/2 \)

\[
\sum_{k=1}^{n} \frac{1}{k} - \ln \left( n + \frac{1}{2} \right) \sim \gamma + \frac{1}{24\left(n + \frac{1}{2}\right)^2} - \frac{7}{960} \left(n + \frac{1}{2}\right)^4 + \frac{31}{8064} \left(n + \frac{1}{2}\right)^6 - \frac{127}{30720} \left(n + \frac{1}{2}\right)^8 + \ldots
\]

(1.6)

and we expect to be much faster than (1.4). Moreover we find the following:
Theorem 1. For every \( n \in \mathbb{N} \), we have
\[
\frac{1}{24\left(n+\frac{1}{2}\right)^2} - \frac{7}{960} + \frac{31}{8064} - \frac{127}{30720} < \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) < \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \frac{7}{960} + \frac{31}{8064}.
\]

The Results. We give the following

Theorem 2. The following standard asymptotic expansion holds as \( n \to \infty \)
\[
\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) - \frac{\gamma}{\pi} \approx \frac{1}{2n} + \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2} B_{2k} \frac{1}{k\pi^2} - \frac{1}{24n^2} - \frac{1}{960n^4} - \cdots
\]

Proof. We have \( \psi(x+1) = H_n - \gamma \) and using (1.3), we get
\[
\sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) = \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) - \ln\left(1 + \frac{1}{2n}\right) - \frac{\gamma}{\pi} \approx \frac{1}{2n} + \sum_{k=2}^{\infty} \frac{B_{2k}}{k\pi^2} \frac{1}{k\pi^2} - \frac{1}{24n^2} - \frac{1}{960n^4} - \cdots
\]

Theorem 1 can be proved by defining the sequences
\[
a_n = \frac{1}{24\left(n+\frac{1}{2}\right)^2} - \frac{7}{960} + \frac{31}{8064} - \frac{127}{30720} - \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right)
\]
\[
b_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) - \left(\frac{1}{24\left(n+\frac{1}{2}\right)^2} - \frac{7}{960} + \frac{31}{8064} - \frac{127}{30720}\right)
\]
and showing that they are strictly increasing to zero. As consequence, \( a_n < 0 \) and \( b_n < 0 \).

REFERENCES