ON AN INEQUALITY FOR THE MEDIANS OF A TRIANGLE

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Abstract. In this paper, we give two new simpler proofs of a sharp inequality for the mediants of a triangle. We also establish two new inequalities by using this sharp inequality. Some related conjectures checked by the computer are put forward, which include two conjectures related to the famous Erdős-Mordell inequality.

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1. INTRODUCTION

In 2000, X.G. Chu and X.Z. Yang [1] established the following geometric inequality: Let $\triangle ABC$ be a triangle with medians $m_a$, $m_b$, $m_c$, circumradius $R$, inradius $r$ and semi-perimeter $s$. Then the following inequality holds:

$$(m_a + m_b + m_c)^2 \leq 4s^2 - 16Rr + 5r^2,$$ (1.1)

with equality if and only if $\triangle ABC$ is equilateral.

This is a strong inequality and has some applications (see e.g. [1], [2]). In my recent paper [3], I have shown that the combinational coefficients in (1.1) is the best possible. In fact, by Theorem 2 in [3] it is easy to prove the following conclusion: For all inequalities in the form

$$(m_a + m_b + m_c)^2 \leq k_1s^2 + k_2Rr + k_3r^2,$$ (1.2)

inequality (1.1) is the best possible, where $k_1$, $k_2$, $k_3$ are constants and satisfy $27k_1 + 2k_2 + k_3 = 81$.

On the other hand, it is interesting that there exists the following sharp inequality (1.3) which is stronger than (1.1):

**Theorem 1.** In any triangle $\triangle ABC$ with sides $a$, $b$, $c$, medians $m_a$, $m_b$, $m_c$, inradius $r$, and circumradius $R$, the following inequality holds:

$$\frac{(m_a + m_b + m_c)^2}{a^2 + b^2 + c^2} \leq 2 + \frac{r^2}{R^2},$$ (1.3)
with equality if and only if triangle ABC is equilateral.

**Remark 1.1.** If \( \Delta ABC \) might be a degenerate triangle, then the equality in (1.3) would also arrive at the case when \( A = 0, B = C = \frac{\pi}{2} \). This fact shows inequality (1.3) is sharp.

H.Y.Yin first posed an equivalent form of (1.3) when the inequality (1.1) just had been set up (see [3], [4]). Until recently, (1.3) has been proved by the author in [3]. However, this proof is very complicated. The author used a lemma in [1], i.e. the inequality:

\[
4m_am_c \leq 2a^2 + bc - \frac{4s(s-a)(b-c)^2}{2a^2 + bc},
\]

with equality if and only if \( b = c \).

In this paper, we give two simpler proofs of Theorem 1, both of which do not depend on (1.4). We also give two applications of Theorem 1. One of them is a beautiful linear inequality involving the medians and the altitudes of a triangle. Another result is about the acute-angled triangle. In the last section, we will propose some related conjectures.

### 2. NEW PROOFS OF THEOREM 1

In this section, we will give two proofs of Theorem 1. To simplify matter, we denote cyclic sums and cyclic products by \( \Sigma, \Pi \) respectively.

**Proof 1:** (The method of \( R - r - s \)) By Cauchy inequality, we have

\[
(\Sigma m_a)^2 \leq (b^2 + c^2)\Sigma \frac{m_a^2}{b^2 + c^2},
\]

i.e.

\[
(\Sigma m_a)^2 \leq 2\Sigma a^2 \Sigma \frac{m_a^2}{b^2 + c^2}. \tag{2.1}
\]

Therefore, to prove inequality (1.3) we need to prove that

\[
\Sigma \frac{m_a^2}{b^2 + c^2} \leq 1 + \frac{r^2}{2R^2}. \tag{2.2}
\]

Using the known formula \( 4m_a^2 = 2(b^2 + c^2) - a^2 \), it is easily known that inequality (2.2) is equivalent to

\[
\Sigma \frac{a^2}{b^2 + c^2} + \frac{2r^2}{R^2} \geq 2. \tag{2.3}
\]

Since
\[
\frac{\sum a^2}{b^2 + c^2} + 3 = \sum \frac{a^2 + b^2 + c^2}{b^2 + c^2} = \sum \frac{1}{b^2 + c^2} = \frac{\sum a^2 \sum (c^2 + a^2)(a^2 + b^2)}{\Pi (b^2 + c^2)} = \frac{\left(\sum a^4 + 3 \sum b^2c^2\right) \sum a^2}{\Pi (b^2 + c^2)},
\]

hence (2.3) is equivalent to
\[
\frac{\left(\sum a^4 + 3 \sum b^2c^2\right) \sum a^2}{\Pi (b^2 + c^2)} + \frac{2r^2}{R^2} - 5 \geq 0.
\]

Thus, we have to prove that
\[
X_1 = R^2 \left(\sum a^4 + 3 \sum b^2c^2\right) \sum a^2 + \left(2r^2 - 5R^2\right) \Pi (b^2 + c^2) \geq 0. \quad (2.4)
\]

Using the following known identities (see e.g. [5]):
\[
abc = 4Rrs, \quad (2.5)
\]
\[
\sum a^2 = 2s^2 - 8Rrs - 2r^2, \quad (2.6)
\]
\[
\sum b^2c^2 = s^4 - 2\left(4R - r\right)s^2r + \left(4R + r\right)^2r^2, \quad (2.7)
\]
\[
\sum a^4 = 2s^4 - 4\left(4R + 3r\right)s^2r + 2\left(4R + r\right)^2r^2, \quad (2.8)
\]
\[
\Pi (b^2 + c^2) = 2s^6 - 2\left(12R - r\right)s^4r + 2\left(40R^2 + 8Rr - r^2\right)s^2r^2 - 2\left(4R + r\right)^3r^3, \quad (2.9)
\]

we obtain
\[
X_1 = 4r^2X_2, \quad (2.10)
\]

where
\[
X_2 = s^6 - \left(8R^2 + 12Rr - r^2\right)s^4 + \left(20R^4 + 32R^3r + 48R^2r^2 + 8R^3 - r^4\right)s^2
\]
\[-\left(4R + r\right)^3r^3 \quad (2.11)
\]

Obviously, the proof of \(X_1 \geq 0\) is changed to \(X_2 \geq 0\). If we put
\[
G_2 = 4R^2 + 4Rr + 3r^2 - s^2,
\]
\[
T_0 = -s^4 + 2\left(2R^2 + 10Rr - r^2\right)s^2 - r(4R + r)^3,
\]

then it is easy to verify the following identity:
\[
X_2 = G_2T_0 + X_3(s^2), \quad (2.12)
\]

where
\[
X_3(s^2) = 2\left(6R + r\right)s^4 + \left(4R^4 - 128R^3r - 84R^2r^2 - 56Rr^3 + 4r^4\right)s^2
\]
\[+ 2\left(2R^2 + 2Rr + r^2\right)(4R + r)^3r.
\]
By identity (2.12), Gerretsen inequality $G_2 \geq 0$ and the fundamental inequality $T_0 \geq 0$ of triangles (see [5], [6]), to prove $X_2 \geq 0$ it remains to prove that

$$X_3\left(s^2\right) \geq 0 \quad (2.13)$$

Let $K \equiv 4(6R + r)s^2 + \left(4R^4 - 128R^3r - 84R^2r^2 - 56Rr^3 + 4r^4\right)$, then it is easy to show that $K$ may be non-negative and also be negative by giving examples. So we can divide the proof of $X_3\left(s^2\right) \geq 0$ into the following two cases, i.e. $K \geq 0$ and $K < 0$.

**Case 1.** Assuming $K \geq 0$.

In this case, according to the property of parabolas and the Gerretsen inequalities:

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \quad (2.14)$$

$X_3\left(s^2\right)$ is strictly increasing on the interval $\left[16Rr - 5r^2, 4R^2 + 4Rr + 3r^2\right]$. So we only need to prove that $X_3\left(16Rr - 5r^2\right) \geq 0$, but

$$X_3\left(16Rr - 5r^2\right) = 2r \left(6R + r\right)\left(16Rr - 5r^2\right)^2 +$$

$$\left(4R^4 - 128R^3r - 84R^2r^2 - 56Rr^3 + 4r^4\right)\left(16Rr - 5r^2\right) +$$

$$2\left(2R^2 + 2Rr + r^2\right)\left(4R + r\right)^3 r$$

$$= 4r\left(80R^3 - 85R^2r + 24Rr^2 + 2r^3\right)\left(R - 2r\right)^2 \geq 0.$$  

The latter inequality follows from Euler inequality $R \geq 2r$. Hence $X_3\left(s^2\right) \geq 0$ is proved under the first case.

**Case 2.** Assuming $K < 0$. For this case, it is easy to know that $X_3\left(s^2\right)$ is decreasing on $\left[16Rr - 5r^2, 4R^2 + 4Rr + 3r^2\right]$. Thus we only need to show $X_3\left(4R^2 + 4Rr + 3r^2\right) \geq 0$. Simple computations give

$$X_3\left(4R^2 + 4Rr + 3r^2\right) = 4\left(4R^4 + 4R^3r + 7R^2r^2 + 4Rr^3 + 2r^4\right)\left(R - 2r\right)^2 \geq 0.$$  

Therefore $X_3\left(s^2\right) \geq 0$ is valid under the second case.

Combining with the arguments of the two cases, $X_3\left(s^2\right) \geq 0$ holds for all triangle $ABC$. Therefore, (2.4), (2.3), (2.2) and (1.3) are all proved. From the deductions above, it is clear that the equality in (1.3) holds only when $\Delta ABC$ is equilateral. The proof of Theorem 1 is complete.

**Proof 2:** (The method of the Difference Substitution) Firstly, we can turn the proof of (1.3) into the inequality (2.3) as above. Since

$$\frac{r}{R} = \frac{\prod\left(b + c - a\right)}{2abc},$$  

$$ (2.15)$$
thus inequality (2.3) is equivalent to
\[
\sum \frac{a^2}{b^2 + c^2} + \frac{\prod (b + c - a)^2}{2(abc)^2} \geq 2,
\]
(2.16)
i.e.,
\[
Y_0 \equiv 2 \left( \frac{abc}{abc} \sum a^2 \left( c^2 + a^2 \right) \left( a^2 + b^2 \right) + \right.
\]
\[
\left. + \prod \left( b^2 + c^2 \right) \prod (b + c - a)^2 - 4 \left( abc \right)^2 \prod (b^2 + c^2) \right) \geq 0.
\]
(2.17)

Let \( b + c - a = 2x \), \( c + a - b = 2y \), \( a + b - c = 2z \), then \( a = y + z \), \( b = z + x \), \( c = x + y \), and we have
\[
Y_0 = 2 \prod (y + z)^2 \sum (y + z)^2 \left[ (x + y)^2 + (y + z)^2 \right] \left[ (y + z)^2 + (z + x)^2 \right]
\]
\[
+ \prod \left[ (z + x)^2 + (x + y)^2 \right] \prod x^2
\]
\[
- 4 \prod (y + z)^2 \prod \left[ (z + x)^2 + (x + y)^2 \right].
\]
(2.18)

Because of symmetry, we assume without loss of generality that \( x \geq y \geq z \) and let
\[
\begin{align*}
y &= z + m \\
x &= z + m + n.
\end{align*}
\]
(2.19)

where \( m \geq 0 \) and \( n \geq 0 \). Substituting (2.19) into (2.18), with help of the mathematical software we obtain the following identity:
\[
Y_0 = 1024 \left( m^2 + mn + n^2 \right)^2 z^8 + 256 \left( 2m + n \right) \left( m^2 + mn + n^2 \right) \left( 13m^2 + 13mn + 6n^2 \right) z^7 +
\]
\[
+ (18816 m^6 + 56448 m^5 n + 79680 m^4 n^2 + 65280 m^3 n^3 + 31296 m^2 n^4 + 8064 mn^5 +
\]
\[
+ 896 n^6)z^6 + 128 \left( 2m + n \right) \left( 118 m^4 + 354 m^3 n + 451 m^2 n^2 + 312 mn^3 + 121 m n^4 +
\]
\[
+ 24 mn^5 + 2 n^6)z^5 + 30112 m^6 + 120448 m^5 n + 202496 m^4 n^2 + 185920 m^3 n^3 +
\]
\[
+ 101792 m^2 n^4 + 3240 m^3 n^5 + 6976 m^2 n^6 + 768 mn^7 + 32 n^8)z^4 +
\]
\[
+ 64 \left( 2m + n \right) \left( 149 m^6 + 447 m^5 n + 513 m^4 n^2 + 281 m^3 n^3 + 79 m^2 n^4 +
\]
\[
+ 13 mn^5 + n^6)z^3 m + 16 \left( 468 m^6 + 1404 m^5 n + 1627 m^4 n^2 + 914 m^3 n^3 + 261 m^2 n^4 +
\]
\[
+ 38 mn^5 + 3 n^6)\left( 2m + n \right) \left( 2m + n \right) \left( 2m + n \right) z^2 m^2 + 16 \left( 13 m^2 + 13 mn + n^2 \right) \left( 2m + n \right)^3 \left( 2m + n \right)^3 zm^3 +
\]
\[
+ 10 \left( 2m + n \right)^4 \left( 2m + n \right)^4 m^4.
\]
(2.20)

So inequality \( Y_0 \geq 0 \) holds obviously by \( m \geq 0 \), \( n \geq 0 \), and \( z > 0 \). Hence (2.17), (2.16) and then (1.3) are proved. The equality in \( Y_0 \geq 0 \) holds if and only if \( m = n = 0 \). Further, it is known that the equalities in (2.17) and (1.3) occurs only when \( a = b = c \), i.e. \( \Delta ABC \) is equilateral. This completes the proof of Theorem 1.

Remark 2.1. From inequality (2.1), using previous methods to prove Theorem 1 we can also prove the following inequality:
which is posed by the author in [3].

3. TWO APPLICATIONS OF THEOREM 1

In this section, we will apply Theorem 1 to establish two new triangle inequalities, which are not both proved by using inequality (1.1).

We first prove the following beautiful linear inequality:

**Theorem A.** For all $\Delta ABC$ holds:

$$m_a + m_b + m_c - (h_a + h_b + h_c) \leq 2(R - 2r),$$  \hspace{1cm} (3.1)

with equality if and only if $\Delta ABC$ is equilateral.

**Proof:** By Theorem 1, to prove (3.1) we need to prove that

$$\left(a^2 + b^2 + c^2\right)\left(2 + \frac{r^2}{R^2}\right) \leq \left[h_a + h_b + h_c + 2(R - 2r)\right]^2.$$  \hspace{1cm} (3.2)

Multiplying both sides of this inequality by $4R^2$ and using the relation $2Rh_a = bc$ etc., inequality (3.2) becomes the following equivalent form:

$$M_0 = \left[bc + ca + ab + 4R \left(R - 2r\right)\right]^2 - 4 \left(a^2 + b^2 + c^2\right)\left(2R^2 + r^2\right) \geq 0.$$  \hspace{1cm} (3.3)

Applying identity (2.6) and the known identity:

$$bc + ca + ab = s^2 + 4Rr + r^2,$$  \hspace{1cm} (3.4)

it is easy to get

$$M_0 = (4R^2 + 4Rr + 3r^2 - s^2)^2.$$

Thus the claimed inequality $M_0 \geq 0$ follows and (3.1) is proved. It is clear that the equality in (3.1) holds only when $\Delta ABC$ is equilateral. This completes the proof of Theorem A.

**Remark 3.1.** By the method to prove Theorem 2 in [3], we can prove that the constant 2 of the right side of (3.1) is the best possible. In addition, from Leuenberger’s inequality (see [6]):

$$h_a + h_b + h_c \leq 2R + 5r,$$  \hspace{1cm} (3.5)

we see that inequality (3.1) is stronger than the known result (see [6]):
\[ m_a + m_b + m_c \leq 4R + r. \]  

**Remark 3.2.** By using inequality (1.1), it is easy to prove another linear inequality for the sum \( m_a + m_b + m_c \):

\[ m_a + m_b + m_c \leq 2s - (6\sqrt{3} - 9)r. \]  

This inequality is also stronger than (3.6) since we have the following inequality:

\[ s \leq 2R + (3\sqrt{3} - 4)r, \]  

which is due to W.J. Blundon (see [7], [8], [9]).

Next, we prove an inequality for the acute-angled triangle, which was found by the author many years ago, but has not been proved before.

**Theorem B.** For acute-angled \( \triangle ABC \) holds:

\[ \frac{h_a + h_b + h_c}{m_a + m_b + m_c} \geq \frac{1}{2} + \frac{r}{R}, \]  

with equality if and only if acute-angled \( \triangle ABC \) is equilateral.

**Proof.** By Theorem 1, to prove (3.9) we need to show that

\[ (h_a + h_b + h_c)^2 - \left( \frac{1}{2} + \frac{r}{R} \right)^2 \left( 2 + \frac{r^2}{R^2} \right) (a^2 + b^2 + c^2) \geq 0. \]  

Multiplying both sides of the above by \( 4R^4 \) and then using the relation \( 2Rh_a = bc \) etc., we see (3.10) is equivalent to

\[ \left( h_a + h_b + h_c \right)^2 - \left( \frac{1}{2} + \frac{r}{R} \right)^2 \left( 2 + \frac{r^2}{R^2} \right) (a^2 + b^2 + c^2) \geq 0. \]  

Substituting (2.6) and (3.4) into the expression of \( N_0 \), then (3.11) is equivalent to

\[ N_0 \equiv s^4R^2 - 4 \left( R^4 + 2R^3r + 4R^2r^2 + 2Rr^3 + 2r^4 \right) s^2 \]
\[ + (4R + r) \left( 4R^4 + 20R^3r + 19R^2r^2 + 8Rr^3 + 8r^4 \right)^2 r \geq 0. \]  

We rewrite \( N_0 \) as follows

\[ N_0 = 4R^2 \left( R + 2r \right) \left( 2R^2 + r^2 \right) e + 8r^4G_2 + R \left[ 4r \left( 3R + r \right) e + RG_1 \right] C_0, \]  

where

\[ e = R - 2r \]
\[ G_1 = s^2 - 16Rr + 5r^2 \]
\[ G_2 = 4R^2 + 4Rr + 3r^2 - s^2 \]
\[ C_0 = s^2 - (2R + r)^2. \]

Therefore, by Euler inequality \( e \geq 0 \), Gerretsen inequalities \( G_1 \geq 0, G_2 \geq 0 \) (see [5], [6]) and the acute triangle inequality \( C_0 \geq 0 \) of Ciamberlini (see [10]), we conclude \( N_0 \geq 0 \) holds for acute-angled \( \Delta ABC \). Hence inequality (3.10) and (3.9) are proved. It is easy to see that the equality in (3.9) holds when \( \Delta ABC \) is equilateral. The proof of Theorem B is completed.

### 4. SEVERAL CONJECTURES

In this section, we will propose some conjectures for the inequalities appeared in this note.

Considering the exponential generalization of Theorem A with help of the computer for verifying, we pose the following three similar conjectures:

**Conjecture 1.** If \( 0 < k < 1 \), then for any \( \Delta ABC \) we have
\[
\left( m_a + m_b + m_c \right)^k - \left( h_a + h_b + h_c \right)^k \leq \left( 2R \right)^k - \left( 4r \right)^k \quad (4.1)
\]

*Remark 4.1.* It is easy to prove that (4.1) is reversed for all triangles if \( k < 0 \).

**Conjecture 2.** If \( \Delta ABC \) is an acute triangle and \( k \geq 1.1 \), then we have
\[
m_a^k + m_b^k + m_c^k - \left( h_a^k + h_b^k + h_c^k \right) \geq 2 \left( R^k - 2^k r^k \right). \quad (4.2)
\]

**Conjecture 3.** If \( k > 1 \) or \( k < 0 \), then for any \( \Delta ABC \) we have
\[
m_a^k + m_b^k + m_c^k - \left( h_a^k + h_b^k + h_c^k \right) \leq \left( 2R \right)^k - \left( 4r \right)^k. \quad (4.3)
\]

When \( k = -1 \), (4.3) is actually equivalent to
\[
\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \leq \frac{1}{2R} + \frac{3}{4r}, \quad (4.4)
\]
which is clearly weaker than the known inequality (see [11]):
\[
\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \leq \frac{1}{2R} + \frac{3}{4r}, \quad (4.5)
\]
where \( w_a, w_b, w_c \) are three internal bisectors of \( \Delta ABC \). On the other hand, (4.4) can be refined as follows:
which is proved by the author in [12].

Considering the lower bound of the left hand side of (3.1), we give

**Conjecture 4.** For any $\Delta ABC$ we have

\[
m_a + m_b + m_c - (h_a + h_b + h_c) \geq s - 3\sqrt{3}r.
\]  

(4.7)

If (4.7) holds true, then Blundon’s inequality (3.8) can be obtain from (3.1) and (4.7).

Next, we give a double inequality conjecture which is inspired by Theorem B:

**Conjecture 5.** For any $\Delta ABC$ we have

\[
\frac{k_a + k_b + k_c}{m_a + m_b + m_c} \geq \frac{1}{2} + \frac{r}{R} \geq \frac{k_a + k_b + k_c}{r_a + r_b + r_c},
\]  

(4.8)

where $k_a, k_b, k_c$ are symmedians of $\Delta ABC$ and $r_a, r_b, r_c$ are radii of excircles of $\Delta ABC$.

Considering the exponential generalization of inequality (2.3), we present

**Conjecture 6.** If $k > 2$, then for any $\Delta ABC$ we have

\[
\frac{a^k}{b^k + c^k} + \frac{b^k}{c^k + a^k} + \frac{c^k}{a^k + b^k} + 2^{k-1} \frac{k^k}{R^k} \geq 2.
\]  

(4.9)

If $0 < k \leq \frac{8}{5}$, then the inequality is reversed.

The classical Erdős-Mordell inequality can be stated as follows: Let $P$ be an interior point of $\Delta ABC$. Denote by $R_1, R_2, R_3$ the distances of $P$ from the vertices $A, B, C$, and $r_1, r_2, r_3$ the distances of $P$ from the sidelines $BC, CA, AB$ respectively. Then holds:

\[
R_1 + R_2 + R_3 \geq 2 (r_1 + r_2 + r_3).
\]  

(4.10)

It is well known that there are a few stronger versions of the Erdős-Mordell inequality (see e.g. [5], [13]). Here, we put forward two new stronger inequalities.

**Conjecture 7.** For any interior point of $\Delta ABC$, we have

\[
\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geq \frac{2 \sqrt{4s^2 - 16Rr + 5r^2}}{m_a + m_b + m_c}
\]  

(4.11)

Inequality (1.1) shows (4.11) is stronger than (4.10).

**Conjecture 8.** For any interior point of $\Delta ABC$, we have
\[
\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geq \frac{2(h_a + h_b + h_c + 2R)}{m_a + m_b + m_c + 4r}.
\] (4.12)

The following equivalent form of (3.1):
\[
h_a + h_b + h_c + 2R \geq m_a + m_b + m_c + 4r
\] (4.13)

means again (4.12) is stronger than the Erdös-Mordell inequality (4.10).

REFERENCES