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SOME CONSEQUENCES OF THE BROWDER’S THEOREM IN HILBERT SPACES

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Abstract. This work deals with the Browder’s theorem in Hilbert spaces. One can remark that the theorems of Riesz, Stampacchia, and Lax-Milgram are consequences of the Browder’s theorem. Moreover in a vector space with a scalar product the Browder’s property is equivalent to the completeness of the space. After that in the non-linear case a Stampacchia and Lax-Milgram type theorems are stated and proved.

Keywords: Browder’s theorem, coercive map, monotone map

1. INTRODUCTION

In this section we present the general setting of this work [1-3].

1.1 Remark. (i). We denote by \(X, Y\) a real dual system, and by \(\langle \cdot, \cdot \rangle_X : X \times Y \to \mathbb{R}\) the corresponding pairing. Moreover \(\sigma(X,Y)\) (respectively \(\sigma(Y,X)\)) denotes the weak topology on \(X\) (respectively on \(Y\)) defined by \(Y\) (respectively \(X\)) [6].

(ii). We assume that \(C\) is a non empty convex, and weakly closed subset of \(X\), and that \(T\) is a (nonlinear) function from \(C\) into \(Y\).

1.2 Definition. (i). We shall say that \(T\) is monotone (respectively strictly monotone) on \(C\) if \(\langle x - y, Tx - Ty \rangle_X \geq 0\) (respectively \(\langle x - y, Tx - Ty \rangle_X > 0\) ) for all \(x, y \in C, x \neq y\).

(ii). The function \(T\) is called weakly continuous on the line segments of \(C\) if the mapping \(\langle c \mapsto \langle x, Tc \rangle : [a,b] \to \mathbb{R}\) is continuous for all \(x \in X\), and \(a, b \in C\).

1.3. Remark. (i). Let \(a \in C\), and \(C - a := \{c - a : c \in C\}\). Obviously \(C - a\) is a convex and weakly closed subset of \(X\) which contains \(0_E\). If we define \(T_a : C - a \to Y, T_a(c - a) := Tc, \forall c \in C\), and if we assume that \(T\) is monotone on \(C\), and weakly continuous on the line segments of \(C\), then the function \(T_a\) has the same properties on \(C\).

(ii). We consider \(E\) a real normed space, and \(E'\) its topological dual endowed with the canonical norm. We shall distinguish the following dual systems. (a). \(\langle E, E' \rangle\) where \(\langle x, x' \rangle_E := x'(x)\); (b). \(\langle E', E \rangle\) where \(\langle x', x \rangle_{E'} := x'(x)\) for all \((x, x') \in E \times E'\).
1.4 Definition. If $X$ is a real normed space, and $b \in X$ then $T$ is called coercive on $C$ with respect to $b$ if

$$\liminf_{c \in C \atop \|c\| \to \infty} \frac{\langle c - b, Tc \rangle}{\|c - b\|} = \infty.$$ 

When $b = 0_E$ we shall say that $T$ is coercive on $C$.

1.5 Remark. In the case of Remark 1.3 (i) we have the following assertions.
(i). If $T$ is coercive on $C$, then $T_a$ is coercive on $C_a$ with respect to $-a$.
(ii). If $T$ is coercive on $C$ with respect to $b$, then $T_a$ is coercive on $C_a$ with respect to $b-a$.

Therefore, from now on, we shall say that $T$ is coercive on $C$ (for example) instead of $T$ is coercive on $C$ with respect to a certain point.

1.6 Definition. Let $X$ be a real normed space.
(i). A function $T$ from $C$ into $Y$ which is monotone on $C$, coercive on $C$ and weakly continuous on the line segments of $C$ is called a Browder-Minty operator on $C$.
(ii). We shall say that a Browder-Minty operator on $C$ has the Browder’s property if it satisfies the following condition,

$$(B). \forall y \in Y, \exists c_y \in C \text{ such that } \langle c_y - c, Tc_y - y \rangle_X \leq 0, \forall c \in C.$$ 

1.7 Theorem. (i). If $E$ is a reflexive Banach space, and $C$ is a subset of $E$ as in Remark 1.1 (ii), then all Browder-Minty operator from $C$ into $E'$ has the Browder’s property (Browder’s theorem, [1, 3]).
(ii). For all Banach space $E$, and $C$ a subset of $E'$ as in Remark 1.1 (ii), all Browder-Minty operator from $C$ into $E$ has the Browder’s property (a Browder type theorem, [7]).

1.8 Remark. (i). When the Browder-Minty operator $T$ is strictly monotone on $C$, the element $c_y$ (where $y \in Y$) is the unique element of $C$ having the (B) - property.
(ii). Since every Hilbert space is a reflexive one we can apply in such a space the Browder’s theorem.

2. LINEAR CONSEQUENCES

From now on $\mathbb{H}$ is a real vector space, and $\langle , \rangle$ is a scalar product on $\mathbb{H}$. As usual $(\mathbb{H}', \|\|)$ denotes its topological dual which is a Banach space with respect to the canonical norm (denoted $\|\|$), and $\langle , \rangle_\mathbb{H}$ is the pairing on $\mathbb{H} \times \mathbb{H}'$.

2.1 Definition. For all $h$ from $\mathbb{H}$ the mapping $\left(x \mapsto \langle x, h \rangle\right): \mathbb{H} \to \mathbb{R}$ is a continuous linear functional on $\mathbb{H}$ with the norm equal to $\|h\|$. We denote by $\mathbb{H}$ the function $\left(h \mapsto \langle h, h \rangle\right): \mathbb{H} \to \mathbb{H}'$, i.e. for all $h \in \mathbb{H}$, the element $\mathbb{H}h \in \mathbb{H}'$ is defined by $\langle x, \mathbb{H}h \rangle_\mathbb{H} = (\mathbb{H}h)(x) := \langle x, h \rangle$. 
2.2 Lema. (i). The function \( \mathbb{I}: (\mathbb{H}, \|\|) \rightarrow (\mathbb{H}', \|\|) \) is an isometric linear operator. 
(ii). The operator \( \mathbb{I} \) is a Browder-Minty strictly monotone operator from \( \mathbb{H} \) into \( \mathbb{H}' \).

Proof: (i). It is obvious.
(ii). Since \( \mathbb{I} \) is a linear operator, and \( \langle h, \mathbb{I}h \rangle_\mathbb{H} = \|h\|^2 \) the assertion is obvious. \( \Box \)

2.3 Theorem. The following assertions are equivalent.
(i). The space \( (\mathbb{H}, \langle , \rangle) \) is a Hilbert space.
(ii) All Browder-Minty operator from a non-empty, convex and closed subset of \( \mathbb{H} \) into \( \mathbb{H}' \) has the Browder’s property.
(iii). The linear operator \( \mathbb{I} \) has the Browder’s property.
(iv). The operator \( \mathbb{I} \) is a bijective isometry.
(v). The Riesz representation theorem is true for \( \mathbb{H}' \) (i.e. for all \( h' \in \mathbb{H}' \) there exists \( h \in \mathbb{H} \) such that \( h'(x) = \langle x, h \rangle \) for all \( x \in \mathbb{H} \)).

Proof: (i) \( \Rightarrow \) (ii). It is the Remark 1.1.8. (ii).
(ii) \( \Rightarrow \) (iii). In view of Lemma 2.2.2 it is obvious.
(iii) \( \Rightarrow \) (iv). By Remark 1.1.8. (i) and the hypothesis, for all \( h' \in \mathbb{H}' \) there exists a unique element \( h \in \mathbb{H} \) such that

\[
\langle h - x, \mathbb{I}h - h' \rangle_\mathbb{H} \leq 0, \ \forall x \in \mathbb{H}.
\]

Since \( \mathbb{H} \) is a vector space this means that

\[
\langle x, \mathbb{I}h - h' \rangle_\mathbb{H} = 0, \ \forall x \in \mathbb{H} \iff \langle h' \rangle = \mathbb{I}h = h'.
\]

(iv) \( \Rightarrow \) (v). It is obvious.
(v) \( \Rightarrow \) (i). By Lemma 2.2.2, and the Riesz representation theorem for \( \mathbb{H}' \) it results directly that \( \mathbb{I}: (\mathbb{H}, \|\|) \rightarrow (\mathbb{H}', \|\|) \) is a bijective, bounded, and linear operator, hence \( \mathbb{I} \) is an isomorphism from \( (\mathbb{H}, \|\|) \) onto \( (\mathbb{H}', \|\|) \). Since \( (\mathbb{H}', \|\|) \) is a Banach space we have that \( (\mathbb{H}, \langle , \rangle) \) is a Hilbert space. \( \Box \)

2.4 Remark. If \( \mathbb{H} \) is a Hilbert space, and \( C \) is a nonempty convex, and closed subset of \( \mathbb{H} \), then \( i_c \) denotes the canonical imbedding of \( C \) into \( \mathbb{H} \). It is obvious that \( i_c \) is a Browder-Minty operator, hence it has the Browder’s property i.e.

\[
\forall h \in \mathbb{H}, \ \exists c_h \in C, \ \langle c_h - c, c_h - h \rangle = \langle c_h - c, i_c c_h - h \rangle \leq 0, \ \forall c \in C.
\]

This means that \( c_h \) is the projection of \( h \) into \( C \) [4].

2.5 Definition. Let \( \varphi \) be a real bilinear form on \( \mathbb{H} \), and \( A \) a non-empty subset of \( \mathbb{H} \). We shall say that \( \varphi \) is coercive on \( A \) if there exists \( \gamma_1 \in (0, \infty) \) such that

\[
\varphi(a, a) \geq \gamma_1 \|a\|^2, \ \forall a \in A.
\]
2.6 Theorem. (Stampacchia’s theorem, [1, 2]). We consider $C$ a non-empty, convex, closed subset of a real Hilbert space $\mathbb{H}$, and $\varphi$ a real bilinear form on the space $(sp_R C) \times \mathbb{H}$ which is coercive on $C$, and separately continuous on $C \times \mathbb{H}$. Then for all $h \in \mathbb{H}$ there exists a unique element $c_h$ from $C$ such that
\[
\varphi(c_h, c_h - c) \leq \langle h, c_h - c \rangle, \quad \forall c \in C.
\]

Proof: For all $c \in C$ the mapping
\[
\left( h \mapsto \varphi(c, h) \right) : \mathbb{H} \to \mathbb{R}
\]
is a linear continuous functional, hence there exists a unique element from $\mathbb{H}$ (denoted by $T_c$) such that
\[
\varphi(c, h) = \langle T_c h, h \rangle, \quad \forall h \in \mathbb{H}
\]
Obviously the mapping $T$ from $C$ into $\mathbb{H}$ is weakly continuous (where $C$ is endowed with the norm topology of $\mathbb{H}$). Moreover for all $c \in C$
\[
\langle Tc, c \rangle = \varphi(c, c) \geq \gamma_1 \| c \|^2,
\]
hence $T$ is coercive on $C$, and for all $c_1, c_2 \in C$
\[
\langle Tc_1 - Tc_2, c_1 - c_2 \rangle = \varphi(c_1 - c_2, c_1 - c_2) \geq \gamma_1 \| c_1 - c_2 \|^2
\]
which means that $T$ is strictly monotone.

In view of the Browder’s property for all $h \in \mathbb{H}$ there exists a unique element $c_h \in C$ such that
\[
\forall c \in C, \langle Tc_h - h, c_h - c \rangle \leq 0 \Leftrightarrow \forall c \in C, \varphi(c_h, c_h - c) = \langle Tc_h, c_h - c \rangle \leq \langle h, c_h - c \rangle.
\]

2.7 Remark. We recall that a real bilinear form, $\varphi$, on $\mathbb{H}$ is continuous if there exists $\gamma_2 \in (0, \infty)$ such that
\[
\forall x, y \in \mathbb{H}, \quad |\varphi(x, y)| \leq \gamma_2 \|x\| \|y\|.
\]

2.8 Theorem. (Lax-Milgram’s theorem, [4]). Let $\mathbb{H}$ be a real Hilbert space, and $\varphi$ a real bilinear, continuous, and coercive form on $\mathbb{H}$. There exists a unique linear, bounded operator $T_\varphi$ on $\mathbb{H}$ such that
\[
\varphi(x, y) = \langle T_\varphi x, y \rangle, \quad \forall x, y \in \mathbb{H}.
\]
Moreover $T_\varphi$ is an invertible operator.

Proof: As in the proof of Theorem 2.2.6 we define $T_\varphi$ a linear operator on $\mathbb{H}$ with the following property
\[
\varphi(x, h) = \langle T_\varphi x, h \rangle, \quad \forall x, h \in \mathbb{H}.
\]
For all \( h \in \mathbb{H} \) we have
\[
\|T_h x\| = \|\varphi(x, \cdot)\| = \sup_{h \in \mathbb{H}, \|h\| \leq 1} \varphi(x, h) \leq \gamma_2 \|x\|
\]
hence \( T_h \) is a bounded linear operator on \( \mathbb{H} \).
Moreover since
\[
\langle T_h x - T_h y, x - y \rangle = \varphi(x - y, x - y) \geq \gamma_1 \|x - y\|^2,
\]
\( T_h \) is a coercive, and strictly monotone operator on \( \mathbb{H} \). Hence, in view of the Browder’s property, \( T_h \) is a surjective map and by the strict monotony it is also injective.
The uniqueness of \( T_h \) is obvious. \( \square \)

3. NON-LINEAR APPLICATIONS

In this section \( \mathbb{H} \) is a real Hilbert space, and \( C \) is a non-empty, convex and closed subset of \( \mathbb{H} \).

3.1 Remark. As in [2] we shall consider from now on a function \( \varphi : C \times \mathbb{H} \rightarrow \mathbb{R} \) which satisfies the following conditions.

\((L_1)\)
(a). For all \( c \in C \), \( \varphi(c, \cdot) \) is a (weakly) continuous linear functional on \( \mathbb{H} \).
(b). For all \( h \in \mathbb{H} \), \( \varphi(\cdot, h) \) is continuous on the line segments of \( C \).

\((L_2)\)
There exists \( \delta \in (0, \infty) \), and \( p \in (1, \infty) \) such that \( \varphi(c, c) \geq \delta \|c\|^p \), \( \forall c \in C \).

\((L_3)\)
For all \( c_1, c_2 \in C \), \( c_1 \neq c_2 \) we have that \( \varphi(c_1, c_1 - c_2) - \varphi(c_2, c_1 - c_2) \geq 0 \).

3.2 Theorem. (a Stampacchia type theorem). For all \( h \in \mathbb{H} \) there exists \( c_h \in C \) such that \( \varphi(c_h, c_h - c) \leq \langle h, c_h - c \rangle \), \( \forall c \in C \).

Proof: Since for all \( c \in C \), \( \varphi(c, \cdot) \in \mathbb{H}' \), there exists a unique element from \( \mathbb{H} \) (denoted by \( Tc \)) with the property \( \forall h \in \mathbb{H} \), \( \varphi(c, h) = \langle Tc, h \rangle \).

In view of the condition \((L_3)\) the function \( T \) from \( C \) into \( \mathbb{H} \) has the following property:
for all \( c_1, c_2 \in C \), \( \langle Tc_1 - Tc_2, c_1 - c_2 \rangle = \varphi(c_1, c_1 - c_2) - \varphi(c_2, c_1 - c_2) \geq 0 \).
i.e. \( T \) is monotone on \( C \).
Moreover by \((L_2)\), for all \( c \in C \) \( \langle Tc, c \rangle = \varphi(c, c) \geq \delta \|c\|^p \),
Hence \( T \) is coercive on \( C \).
According to the condition \((L_1)\) \((b)\)
\[
(c \mapsto \langle Tc, h \rangle) = (c \mapsto \varphi(c, h)) : C \rightarrow \mathbb{R}
\]
is continuous on the line segments of \( C \), and this means that \( T \) is weakly continuous on the line segments of \( C \).
Therefore, \( T \) is a Browder-Minty operator from \( C \) into \( \mathbb{H} \), hence it has the Browder’s property, i.e. \( \forall h \in \mathbb{H}, \exists c_h \in C \) such that \( \langle Tc_h - h, c_h - c \rangle \leq 0 \), \( \forall c \in C \).
We have that
\[ \varphi(c_h, c_h - c) = \langle Tc_h, c_h - c \rangle \leq \langle h, c_h - c \rangle, \quad \forall c \in C. \]

**3.3 Corollary.** If the inequality from the condition (L3) is a strict one, then for all \( h \in \mathbb{H} \) the element \( c_h \) from \( C \) which satisfies the relation of the previous theorem is unique.

**Proof:** It is obvious because in this case \( T \) is strictly monotone. \( \square \)

**3.4 Theorem.** (a Lax – Milgram type theorem). Let \( C \) be equal to \( \mathbb{H} \). Then there exists a unique (nonlinear) Browder-Minty operator from \( \mathbb{H} \) into \( \mathbb{H} \) such that
\[ \varphi(x, y) = \langle Tx, y \rangle, \quad \forall x, y \in \mathbb{H}. \]

**Proof:** We apply the same steps as in the proof of the Theorem 3.3.2. \( \square \)

**3.5 Corollary.** Suppose that the inequality (L3) is strict. Then the operator \( T \) of the previous theorem is a bijective function.

**Proof:** By the Browder’s property it is surjective, and in view of our hypothesis \( T \) is injective. \( \square \)

**REFERENCES**