A CLASS OF FUNCTIONAL EQUATIONS FOR INVOLUTIVE AUTOMORPHISMS OF $n$–GROUPS

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Abstract. In a $n$-group $(G, [\,\,])$ consider the following functional equations:

$$(E_p): \quad f([x_1,\ldots,x_p, f(x_{p+1}),\ldots, f(x_n)]) = [f(x_1),\ldots, f(x_p), x_{p+1},\ldots,x_n]$$

$(x_1,\ldots,x_p, x_{p+1},\ldots,x_n \in G, \, 1 \leq p \leq n - 1)$. In the case of groups these equations were studied by I. Corovei and V. Pop [4]. We study this class of equations and characterize their solutions using involutive automorphisms of $n$-groups.

Keywords: functional equation, automorphisms, $n$-groups.

1. INTRODUCTION

On of the most efficient tool in the theory of $n$-groups is the reducing method, in order to use known results from group theory. By Hosszú theorem [3], we associate to a $n$-group a family of reduced groups, all of these giving by extension, the initial $n$-group.

In this paper we will use these methods and we reveal the necessary results and notions.

Let $(G, [\,\,])$ be a $n$-group with the $n$-ary operation $[\,\,] : G^n \rightarrow G$ and let us denote by $\bar{e}$ the skew element of $e \in G$.

For every $e \in G$ we define the binary operation on $G$ by

$$x \cdot y = [x, e, e, \bar{e}, y], \quad x, y \in G. \quad (2.1)$$

The pair $(G, \cdot)$ is a group, which is called the reduced group in Hosszú sense, and we denote $(G, \cdot) = \text{Red}_e(G, [\,\,])$.

M. Hosszú [3] has proved that the function

$$\alpha_e : G \rightarrow G, \quad \alpha_e(x) = [e, x, e, e, \bar{e}], \quad x \in G, \quad (2.2)$$

is an automorphism of $(G, \cdot), \quad \alpha_e^{-1}$ is an inner automorphism

$$\alpha_e^{-1}(x) = a \cdot x \cdot a^{-1}, \quad x \in G, \text{ where } a = [e]. \quad (2.3)$$

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Conversely: If \((G,\cdot)\) is a group, for every pair \((\alpha,a)\), where \(a \in G\), \(\alpha\) is an automorphism of \((G,\cdot)\), \(\alpha^{n-1}(x) = a \cdot x \cdot a^{-1}, \ x \in G\), then the \(n\)-ary operation

\[
[x_1,\ldots,x_n] = x_1 \cdot \alpha(x_2) \cdot \ldots \cdot \alpha^{n-1}(x_n) \cdot a
\]  

(2.4)
determines on \(G\) a \(n\)-group structure that is called \(n\)-ary extension of \((G,\cdot)\) in Hosszú sense and is denoted by

\[
(G,[\ ] = \text{Ext}_{\alpha,a}(G,\cdot).
\]

M. Hosszú [3] show that

\[
\text{Ext}_{\alpha,a}(G,\cdot) = (G,[\ ])
\]

(2.5)
(every \(n\)-ary operation which determines a \(n\)-group is of the form (1.4)).

The relations between the morphisms of \(n\)-groups and its reduced group was established in [1, 2].

**Theorem.** [2] A map \(f : G \to G\) is a morphism of \(n\)-group \((G,[\ ]) = \text{Ext}_{\alpha,a}(G,\cdot)\) iff there exist a binary morphism of groups \((G,\cdot), \ g : G \to G\) such that:

a) \(f(x) = f(e) \cdot g(x), \ x \in G\);

b) \(g(\alpha(x)) = \alpha(f(e)) \cdot \alpha(g(x)) \cdot (\alpha(f(e)))^{-1};\)

c) \(f(a) = [f(e)].\)

(2.6)

We recall that a function \(f : G \to G\) is involutive if \(f \circ f = 1_G\)
or \(f = f^{-1}.\)

(2.7)

**2. THE FUNCTIONAL EQUATIONS** \(f : G \to G\)

\((E_p) : \quad f([x_1,\ldots,x_p,f(x_{p+1}),\ldots,f(x_n)]) = [f(x_1),\ldots,f(x_p),x_{p+1},\ldots,x_n]\)

**ON A \(n\)-GROUP \((G,[\ ])\)**

Let \((G,[\ ])\) be a \(n\)-group and \(1 \leq p \leq n - 1\). We consider the functional equation on \(G:\)

\[
(E_p) \quad \begin{cases}
    f : G \to G \\
    f([x_1,\ldots,x_p,f(x_{p+1}),\ldots,f(x_n)]) = [f(x_1),\ldots,f(x_p),x_{p+1},\ldots,x_n]
\end{cases}
\]

\(x_1,\ldots,x_p,x_{p+1},\ldots,x_n \in G.\)

For \(e \in G\) we consider the Hosszú reduced group \((G,\cdot) = \text{Red}_e(G,[\ ])\) using the same notation as (1.1), (1.2), (1.4).
Theorem 2.1. If the function \( f : G \to G \) verifies the equation \((E_p)\) in the \( n \)-group \((G,[\ ]))\) then \( f \) verifies the equation \((2.1)\) on the group \((G,\cdot)\)

\[
f(x \cdot b \cdot f(y)) = f(x) \cdot c \cdot y, \ x,y \in G
\]  
(2.1)

where \( b = \alpha^n(f(e)) \cdot \alpha^{n+1}(f(e)) \cdot \ldots \cdot \alpha^{n-2}(f(e)) \cdot a, \ c = \alpha(f(e)) \cdot \alpha^2(f(e)) \cdot \ldots \cdot \alpha^{n-1}(f(e)) \cdot a. \)

Proof: Taking in \((E_p)\) \( x_1 = x, \ x_2 = \ldots = x_{n-1} = e, \ x_n = y \) and using \( \alpha(e) = \ldots = \alpha^{n-1}(e) = e, \ \alpha(\overline{e}) = \overline{e}, \ \alpha^{n-1}(x) = a \cdot x \cdot a^{-1} \) we obtain the equation \((2.1)\). □

Theorem 2.2. If the function \( f \) verifies the equation \((2.1)\) then \( f \) is a bijection and the function \( G : G \to G \) defined by \( f(x) = g(y^{-1} \cdot x), \ x \in G, \) satisfies the equation

\[
g(u \cdot g(v)) = g(u) \cdot v, \ u,v \in G,
\]  
(2.2)

where \( y_0 \in G \) satisfies \( f(y_0) = e. \)

Proof: Taking in \((2.1)\) \( a = b^{-1} \) it follows \( f(f(y)) = f(b^{-1}) \cdot c \cdot y, \ y \in G \) and since the translation \( h : G \to G, \ h(y) = f(b^{-1}) \cdot c \cdot y, \ y \in G \) is a bijection it follows that \( f \) is bijection too.

Let now \( y = y_0 \) in \((1)\) where \( f(y_0) = e. \)

It follows \( f(x \cdot b) = f(x) \cdot c \cdot y_0, \ x \in G. \)

The equation \((2.1)\) becomes \( f(xbf(y)) = f(xb) \cdot y_0^{-1} \cdot y, \ x,y \in G. \) By the transformation \( f(x) = g(y^{-1} \cdot x), \ x \in G \) we obtain: \( g(y_0^{-1} \cdot xb \cdot g(y^{-1} \cdot y)) = g(y_0^{-1} \cdot xb)_0^{-1} \cdot y, \ x,y \in G. \) Denoting \( u = y_0^{-1} \cdot x \cdot b, \ v = y_0^{-1} \cdot y \) it follows: \( g(u \cdot g(v)) = g(u) \cdot v, \) for every \( u,v \in G. \) □

Theorem 2.3. The function \( G : G \to G \) satisfies the equation \((2.2)\) iff \( g \) is an involutive automorphism of the group \((G,\cdot).\)

Proof: Taking in \((2.2)\) \( u = e \) it follows \( g(g(v)) = g(e) \cdot v, \ v \in G, \) so \( g \) is a morphism. For \( u = v = e \) it follows \( g(g(e)) = g(e), \) therefore \( g(e) = e \) and \( g(g(v)) = v, \ v \in G \) \((g \) is idempotent). The equation \((2.2)\) becomes: \( g(u \cdot g(v)) = g(u) \cdot g(g(v)), \ u,v \in G \) or \( g(u \cdot t) = g(u) \cdot g(t), \) for all \( u \in G, \ t = g(v) \in G, \) so \( g \) is a morphism.

Remark. The solutions of the functional equation

\[
g : \mathbb{R} \to \mathbb{R}, \ g(x + g(y)) = g(x) + y, \ x,y \in \mathbb{R}
\]

are the additive functions which on a Hamel basis \( H \) are defined as follows:

Let \( H \) be partitioned as \( H = H_0 \cup H_1 \cup H_2 \) and suppose that there exists a bijection \( \varphi : H_1 \to H_2. \)

We define \( f(h_0) = h_0, \ h_0 \in H_1, \ f(h_1) = \varphi(h_1), \ h_1 \in H_1, \ f(h_2) = \varphi^{-1}(h_2), \ h_2 \in H_2. \)
Theorem 2.4. The function \( f \) satisfies the equation (2.1) iff 
\[ f(x) = d \cdot g(x), \ x \in G, \] 
where \( d \in G, \ g : G \to G \) is an involutive automorphism of \((G, \cdot)\) and \( g(b \cdot d) = c, \ b, c \) are defined in Theorem 2.1.

**Proof:** From Theorem 2.2 and Theorem 2.3 we have 
\[ f(x) = g(v_0^{-1} \cdot x) = g(v_0^{-1}) \cdot g(x) = d \cdot g(x), \ x \in G, \] 
where \( d = g(v_0^{-1}) = f(e) \) and \( g \) is an idempotent automorphism. Taking account of (2.1) we obtain: 
\[ dg(xbdg(y)) = dg(x)c_y \text{ or } dg(x)g(bdg(g(y))) = dg(x)c_y \text{ or } g(bd) = c. \]

Theorem 2.5. If the function \( f \) satisfies the equation \((E_p)\) on the \( n \)-group \((G, [\cdot])\) then there exists an element \( d \in G \), an automorphism \( g \) of bigroup \((G, \cdot) = \text{Red}_e(G, [\cdot])\) such that:

a) \( f(x) = d \cdot g(x), \ x \in G; \)

b) \( g(g(x)) = x, \ x \in G; \)

c) \( f([e, d]) = [d, e]. \)

**Proof:** Using Theorems 2.1, 2.2 and 2.3 it follows a) and b).
The relation c) is 
\[ f([e, f(e)]) = [f(e), e] \] 
which is the same with \( g(b \cdot d) = c \) from Theorem 2.4.

Remark. Taking in \((E_p)\) \( x_1 = e, \ x_2 = x, \ x_3 = \ldots = x_n = e \) we can prove the relation: 
\[ g(\alpha(x)) = \alpha(f(e)) \cdot \alpha(g(x)) \cdot (\alpha(f(e)))^{-1}, \ x \in G, \] 
which is a necessary condition (Theorem 1.6), that \( f \) to be morphism of \( n \)-group \((G, [\cdot])\), but it is not sufficiently (the relation c) of (1.6) is not verified). This is true if \( f(e) = e \).

Theorem 2.6. If the function \( f : G \to G \) has a fixed point, then the only solutions of equation \((E_p)\) are the involutive automorphisms of \( n \)-group \((G, [\cdot])\).

**Proof:** Choosing \( e \in G \) a fixed point of \( f \), from Theorem 2.5 we have \( d = e \) and \( f = g \). From Remark and [5] it follows that \( f \) is an automorphism of \( n \)-group \((G, [\cdot])\).

Conversely, if \( f \) is an automorphism of \( n \)-group with the property \( f(f(x)) = x, \ x \in G, \) then \( f \) verifies the equation \((E_p)\).

REFERENCES