THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A TIMELIKE CURVE IN MINKOWSKI 4-SPACE

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Abstract. In this study, some interesting results about the original timelike curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T(S^3)$ and $T(H^3)$. Keywords: Natural Lift, Geodesic Sprays.

1. INTRODUCTION AND PRELIMINARIES

In differential geometry, there are many important consequences and properties of curve [2, 3, 14] Thorpe gave the concepts of the natural lift curve and geodesic spray in [9]. Thorpe proved the natural lift $\tilde{\alpha}$ of the curve $\alpha$ is an integral curve of the geodesic spray iff $\alpha$ is an geodesic on $M$ in [9]. Çalışkan, Sivridağ and Hacısalıoğlu studied the natural lift curves of the spherical indicatrices of tangent, principal normal, binormal vectors and fixed centrode of a curve in [12]. They gave some interesting results about the original curve were obtained, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T(S^3)$ in [12]. The analogue of the theorem of Thorpe was given in Minkowski 3-space and in Minkowski 4-space by Ergün and Çalışkan in [4,5]. Çalışkan and Ergün defined $\tilde{M}$-vector field $Z$, $\tilde{M}$-geodesic spray, $\tilde{M}$-integral curve of $Z$, $\tilde{M}$-geodesic in [13]. The analogue of the theorem of Sivridağ and Çalışkan was given in Minkowski 3-space by Ergün and Çalışkan in [13]. At each point of a differentiable curve a tetrad of mutually orthogonal unit vectors (called tangent, normal, binormal, and trinormal) was defined and constructed. The rates of changes of these vectors along the curve...
define the curvatures of the curve in the space $E^4_1$. Spherical images (indicatrices) are a well-known concept in classical differential geometry of curves [16].

To meet the requirements in the next sections, here the basic elements of the theory of curves in the space $E^4_1$ are briefly presented. (A more complete elementary treatment can be found in [10].)

Let Minkowski 4-space $\mathbb{IR}^4_1$ be the vector space $\mathbb{IR}^4$ equipped with the Lorentzian inner product $g$ given by

$$g(X, X) = -x_1^2 + x_2^2 + x_3^2 + x_4^2$$

where $X = (x_1, x_2, x_3, x_4) \in \mathbb{IR}^4$. A non-zero vector $X = (x_1, x_2, x_3, x_4) \in \mathbb{IR}^4$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ and lightlike (or null) if $g(X, X) = 0$. The norm of a vector $X$ is defined by

$$\|X\|_g = \sqrt{g(X, X)}.$$

We denote by $\{T(t), N(t), B_1(t), B_2(t)\}$ the moving Frenet frame along the curve $\alpha$. Let $\alpha$ be a unit speed timelike space curve. The functions $\kappa_1$, $\kappa_2$ and $\kappa_3$ are called the first, second and third curvature of $\alpha$. Let Frenet vector fields of $\alpha$ be $\{T, N, B_1, B_2\}$. $N$ is spacelike vector field. In this trihedron, we assume that $T$ and $B_1$ are spacelike vector fields and $B_2$ is a timelike vector field. Then, Frenet formulas are given by

$$T = \kappa_1 N, \quad \dot{N} = \kappa_1 T + \kappa_2 B_1, \quad \dot{B}_1 = -\kappa_2 N + \kappa_3 B_2, \quad B_2 = -\kappa_3 B_1, \quad [10].$$

Let $\alpha$ be any timelike curve with $\alpha = \alpha(t)$.

**Type I. General Case:** $\kappa_1 \neq 0 \quad \kappa_2 \neq 0$ and $\kappa_3 = 0$. The helix is not confined to any 3-flat in space-time [17].

**Type II. Degenerate:** $\kappa_1 \neq 0 \quad \kappa_2 \neq 0$ and $\kappa_3 \neq 0$. The helix lies in a 3-flat [17].

**Subtype II$_a$:** $\kappa_1^2 - \kappa_2^2 > 0$

**Subtype II$_b$:** $\kappa_1^2 - \kappa_2^2 = 0$

**Subtype II$_c$:** $\kappa_1^2 - \kappa_2^2 < 0$.

**Type III. Degenerate:** $\kappa_1 = 0 \quad \kappa_3 = 0$ and $\kappa_3 = 0$. The helix lies in a 2-flat: It is a pseudocircle, a curve of hyperbolic type, [17].
**Type IV. Degenerate:** \( \kappa_1 = 0 \quad \kappa_2 = 0 \) and \( \kappa_3 = 0 \). The helix is a straight line, [17].

Let \( M \) be a hypersurface in \( \mathbb{R}^4 \) and let \( \alpha : I \to M \) be a parametrized curve. \( \alpha \) is called an integral curve of \( X \) if

\[
\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \quad \text{(for all } t \in I )
\]

where \( X \) is a smooth tangent vector field on \( M \), [3]. We have

\[
TM = \bigcup_{P \in M} T_P M = \chi(M)
\]

where \( T_P M \) is the tangent space of \( M \) at \( P \) and \( \chi(M) \) is the space of vector fields of \( M \).

**Definition 1.** For any parametrized curve \( \alpha : I \to M \), \( \overline{\alpha} : I \to TM \) given by

\[
\overline{\alpha}(t) = \left[ \alpha(t), \dot{\alpha}(t) \right] = \alpha(t)_{\alpha(t)}
\]

is called the natural lift of \( \alpha \) on \( TM \), [6]. Thus, we can write

\[
\frac{d\overline{\alpha}}{dt} = \frac{d}{dt} \left( \alpha(t)_{\alpha(t)} \right) = D_{\alpha(t)} \dot{\alpha}(t)
\]

where \( D \) is the Levi-Civita connection on \( \mathbb{R}^4 \).

Let \( \alpha \) be a unit speed timelike space curve. Then the natural lift \( \overline{\alpha} \) of \( \alpha \) is a spacelike space curve, [6].

**Definition 2.** A \( X \in \chi(TM) \) is called a geodesic spray if for \( V \in TM \)

\[
X(V) = e g(S(V), V) N
\]

where \( e = g(N, N) \), [6].

**Theorem 1.** The natural lift \( \overline{\alpha} \) of the curve \( \alpha \) is an integral curve of geodesic spray \( X \) if and only if \( \alpha \) is a geodesic on \( M \), [6].
2. THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A TIMELIKE CURVE IN MINKOWSKI 4-SPACE

Let \( D, \overline{D} \) and \( \ddot{D} \) be connections in \( \mathbb{R}^4 \), \( S^3 \) and \( H^3 \) respectively and \( \xi \) be a unit normal vector field of \( S^3 \) and \( H^3 \). Then Gauss Equations are given by the followings

\[
\begin{align*}
D_X Y &= \overline{D}_X Y + \varepsilon g(S(X),Y)\xi, \\
D_X Y &= \ddot{D}_X Y + \varepsilon g(S(X),Y)\xi,
\end{align*}
\]

where and \( \varepsilon = g(\xi,\bar{\xi}) \) and \( S \) is the shape operator of \( S^3 \) and \( H^3 \) and

\[ S(X) = -D_X \bar{\xi}. \]

Let \( \alpha \) be a unit speed timelike space curve, \( T \) is timelike vector field, \( N, B_1 \) and \( B_2 \) are spacelike vector fields.

2.1. The Natural Lift of the Spherical Indicatrix of Tangent Vector of \( \alpha \)

Let \( \alpha_T \) be the spherical indicatrix of tangent vectors of \( \alpha \) and \( \overline{\alpha_T} \) be the natural lift of the curve \( \alpha_T \). If \( \overline{\alpha_T} \) is an integral curve of the geodesic spray, then from Theorem 1, we have

\[
\overline{D}_{\alpha_T} \overline{\alpha_T} = 0
\]

that is

\[
D_{\alpha_T} \alpha_T = \overline{D}_{\alpha_T} \alpha_T + \varepsilon g\left(S\left(\alpha_T\right),\dot{\alpha_T}\right)\bar{\xi}
\]

\[
D_{\alpha_T} \alpha_T = \varepsilon g\left(S\left(\alpha_T\right),\dot{\alpha_T}\right)T
\]

\[
D_{\alpha_T} \alpha_T = D_{\alpha_T} \left(\kappa_1 N + \kappa_1 (\kappa_T + \kappa_2 B)\right) \frac{ds}{ds_T},
\]

\[
\frac{ds}{ds_T} = \frac{1}{\kappa_1}
\]

\[
D_{\alpha_T} \alpha_T = \left(\kappa_1 N + \kappa_1 (\kappa_T + \kappa_2 B)\right) \frac{1}{\kappa_1}
\]

and

\[ \varepsilon = g(\bar{\xi},\bar{\xi}) = g(T,T) = -1, \]

\[
g\left(S\left(\alpha_T\right),\dot{\alpha_T}\right) = -\kappa_1^2.
\]
Using these in the Gauss equation, we immediately have

\[ \ddot{D}_{\alpha r} \alpha_r = \left( \kappa_1 N + \kappa_1^2 \left( \kappa_1 T + \kappa_2 B_1 \right) \right) \frac{1}{\kappa_1^2} - \kappa_2^2 T \]

From the Eq. (2.3) we get

\[ \left( \frac{\kappa_1^2}{\kappa_1} - \kappa_1^2 \right) T + \left( \frac{\kappa_1}{\kappa_1} \right) N + \left( \frac{\kappa_1 \kappa_2}{\kappa_1} \right) B_1 = 0 \]

Since \( T, N, B_1, B_2 \) are linearly independent we obtain

\[
\begin{align*}
\frac{\kappa_1}{\kappa_1} - \kappa_1^2 &= 0 \\
\frac{\kappa_1}{\kappa_1} &= 0 \\
\frac{\kappa_1 \kappa_2}{\kappa_1} &= 0
\end{align*}
\]

So from the Eq. (2.7) we can give the following proposition.

**Corollary 1.** If the natural lift \( \overline{\alpha} \) of \( \alpha \) is an integral curve of the geodesic on the tangent bundle \( T \left( S^3 \right) \), then \( \alpha \) can be helices of Type III.

### 2.2. The Natural Lift of the Spherical Indicatrix of Principal Normal Vector of \( \alpha \)

Let \( \alpha_N \) be the spherical indicatrix of tangent vectors of \( \alpha \) and \( \overline{\alpha}_N \) be the natural lift of the curve \( \alpha_N \). If \( \overline{\alpha}_N \) is an integral curve of the geodesic spray, then because of **Theorem 1.** we have

\[ \tilde{D}_{\alpha_N} \dot{\alpha}_N = 0 \]

that is

\[ D_{\alpha_N} \dot{\alpha}_N = \tilde{D}_{\alpha_N} \dot{\alpha}_N + \varepsilon \left( S \left( \dot{\alpha}_N, \dot{\alpha}_N \right), \dot{\alpha}_N \right) \xi \]

\[ D_{\alpha_N} \dot{\alpha}_N = \varepsilon \left( S \left( \dot{\alpha}_N, \dot{\alpha}_N \right), \dot{\alpha}_N \right) N \]
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\[ D_{\alpha_N} \ddot{\alpha}_N = D_{\alpha_N} \left( \kappa_1 T + \kappa_2 B_1 \right) \]
\[ = \left( \kappa_1 T + \kappa_1 \left( \kappa_1 N + \kappa_2 B_1 + \kappa_2 \left( -\kappa_2 N + \kappa_1 B_2 \right) \right) \right) \frac{ds}{ds_N} , \]
\[ \frac{ds}{ds_N} = \frac{1}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \]
\[ D_{\alpha_N} \ddot{\alpha}_N = \left( \kappa_1 T + \kappa_1 \left( \kappa_1 N + \kappa_2 B_1 + \kappa_2 \left( -\kappa_2 N + \kappa_1 B_2 \right) \right) \right) \frac{1}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \]
and
\[ \varepsilon = g(\xi, \xi) = g(N, N) = 1, g \left( S \left( \alpha_N \right), \dot{\alpha}_N \right) = \kappa_1^2 - \kappa_2^2 . \]

Using these in the Gauss equation, we immediately have
\[ \ddot{D}_{\alpha_N} \alpha_N = \left( \kappa_1 T + \left( \kappa_1^2 - \kappa_2^2 \right) N + \kappa_1 \kappa_2 B_2 + \kappa_2 B_1 \right) \frac{1}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \]
\[ + \left( -\kappa_1^2 + \kappa_2^2 \right) N \]
From the Eq. (2.8) we get
\[ \left( \frac{\kappa_1}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \right) T + \left( \frac{\kappa_1^2 - \kappa_2^2}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} + \left( -\kappa_1^2 + \kappa_2^2 \right) \right) N \]
\[ + \left( \frac{\kappa_2}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \right) B_1 + \left( \frac{\kappa_1 \kappa_2}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \right) B_2 = 0 \]
Since \( T, N, B_1, B_2 \) are linearly independent we obtain ,
\[ \left\{ \begin{array}{l}
\left( \frac{\kappa_1}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} \right) = 0 \\
\frac{\kappa_1^2 - \kappa_2^2}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} + \left( -\kappa_1^2 + \kappa_2^2 \right) = 0 \\
\frac{\kappa_1}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} = 0 \\
\frac{\kappa_1 \kappa_2}{\sqrt{(-\kappa_1^2 + \kappa_2^2)}} = 0
\end{array} \right. \]
So from the Eq. (2.12) we can give the following proposition.

**Corollary 2.** If the natural lift $\overline{\alpha_N}$ of $\alpha_N$ is an integral curve of the geodesic on the tangent bundle $T(S^1)$, then $\alpha$ can be helices of Type II (Subtype II_a or Subtype II_c).

### 2.3. The Natural Lift of the Spherical Indicatrix of the First Binormal Vectors of $\alpha$

Let $\alpha_{B_i}$ be the spherical indicatrix of tangent vectors of $\alpha$ and $\overline{\alpha_{B_i}}$ be the natural lift of the curve $\alpha_{B_i}$. If $\overline{\alpha_{B_i}}$ is an integral curve of the geodesic spray, then by using Theorem 1, we have

$$\widetilde{D}_{\overline{\alpha_N}} \overline{\alpha_N} = 0$$

that is

$$D_{\alpha_N} \overline{\alpha_N} = \widetilde{D}_{\overline{\alpha_N}} \overline{\alpha_N} + \varepsilon g \left( S \left( \overline{\alpha_N} , \overline{\alpha_N} \right) , \overline{\alpha_N} \right) \xi$$

$$D_{\alpha_N} \overline{\alpha_N} = \varepsilon g \left( S \left( \overline{\alpha_N} , \overline{\alpha_N} \right) , \overline{\alpha_N} \right) B_1$$

$$D_{\alpha_N} \overline{\alpha_N} = \frac{ds}{ds_{B_i}} = \frac{1}{\sqrt{\left( \kappa_2^2 + \kappa_3^2 \right)}}$$

$$D_{\alpha_N} \overline{\alpha_N} = \frac{1}{\sqrt{\left( \kappa_2^2 + \kappa_3^2 \right)}}$$

and

$$\varepsilon = g (\xi , \overline{\xi}) = g (B_1 , B_1) = 1 , g \left( S \left( \overline{\alpha_N} , \overline{\alpha_N} \right) , \overline{\alpha_N} \right) = - \left( \kappa_2^2 + \kappa_3^2 \right)$$

Using these in the Gauss equation, we immediately have

$$\widetilde{D}_{\overline{\alpha_N}} \overline{\alpha_N} = \left( - \kappa_2 \kappa_3 + \kappa_3 B_2 \right) \overline{\alpha_N}$$

From the Eq. (2.13) we get
The natural lift curve of $\alpha_{B_1}$ is an integral curve of the geodesic on the tangent bundle $T(S^3_1)$, then $\alpha$ can be helices of Type II or helices of Type III.

**Corollary 3.** If the natural lift $\alpha_{B_1}$ of $\alpha_{B_1}$ is an integral curve of the geodesic on the tangent bundle $T(S^3_1)$, then $\alpha$ can be helices of Type II or helices of Type III.

### 2.4. The Natural Lift of the Spherical Indicatrix of the Second Binormal Vectors of $\alpha$

Let $\alpha_{B_1}$ be the spherical indicatrix of tangent vectors of $\alpha$ and $\alpha_{B_2}$ be the natural lift of the curve $\alpha_{B_1}$. If $\alpha_{B_2}$ is an integral curve of the geodesic spray, then by using Theorem 1, we have

$$\hat{D}_{\alpha_{B_2}} \dot{\alpha}_{B_2} = 0$$

that is

$$D_{\alpha_{B_2}} \dot{\alpha}_{B_2} = \hat{D}_{\alpha_{B_2}} \dot{\alpha}_{B_2} + \varepsilon g \left( S \left( \dot{\alpha}_{B_2}, \dot{\alpha}_{B_2} \right), \dot{\alpha}_{B_2} \right) \hat{B}_2$$

$$D_{\alpha_{B_2}} \dot{\alpha}_{B_2} = \varepsilon g \left( S \left( \dot{\alpha}_{B_2}, \dot{\alpha}_{B_2} \right), \dot{\alpha}_{B_2} \right) B_2$$
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\[ D_{\kappa_1} \dot{\alpha}_{\kappa_2} = D_{\kappa_1} \left( -\kappa_3 B_1 \right) = \left( -\kappa_3 B_1 - \kappa_3 \left( -\kappa_2 N + \kappa_3 B_2 \right) \right) \frac{ds}{ds_{\kappa_3}}, \]
\[ \frac{ds}{ds_{\kappa_3}} = \frac{1}{\kappa_3} \]
\[ D_{\kappa_2} \dot{\alpha}_{\kappa_2} = \left( -\kappa_3 B_1 - \kappa_3 \left( -\kappa_2 N + \kappa_3 B_2 \right) \right) \frac{1}{\kappa_3} \]

and
\[ \varepsilon = g \left( \xi, \xi \right) = g \left( B_2, B_2 \right) = 1, \quad \left( S \left( \dot{\alpha}_{\kappa_2}, \dot{\alpha}_{\kappa_2} \right) \right) = \kappa_3^2. \]

Using these in the Gauss equation, we immediately have
\[ \tilde{D}_{\kappa_2} \dot{\alpha}_{\kappa_2} = \left( -\kappa_3 B_1 - \kappa_3 \kappa_2 N + \kappa_3^2 B_2 \right) \frac{1}{\kappa_3} - \kappa_3^2 B_2 \]

From the Eq. (2.18) we get
\[ \left( -\frac{\kappa_3 \kappa_2}{\kappa_3} \right) N + \left( -\frac{\kappa_3}{\kappa_3} \right) B_1 + \left( \frac{\kappa_3^2}{\kappa_3} - \kappa_3^2 \right) B_2 = 0 \]

Because \( T, N, B_1, B_2 \) are linearly independent, we have,
\[ \begin{cases} -\frac{\kappa_3 \kappa_2}{\kappa_3} = 0 \\ -\frac{\kappa_3}{\kappa_3} = 0 \\ \frac{\kappa_3^2}{\kappa_3} - \kappa_3^2 = 0 \end{cases} \]

So from the Eq. (2.22) we can give the following proposition.

**Corollary 4.** If the natural lift \( \overline{\alpha}_{\kappa_2} \) of \( \alpha_{\kappa_2} \) is an integral curve of the geodesic on the tangent bundle \( T \left( H_0^3 \right) \) then we have \( \kappa_2 = 0 \) and \( \kappa_3 = 1 \). Therefore there is no such curve satisfying this condition.
REFERENCES