ON NANO SEMI ALPHA OPEN SETS
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Abstract. In this paper, we presented another concept of N-O.S. called NSₐα-O.S. and studied their fundamental properties in nano topological spaces. We also present NSₐα-interior and NSₐα-closure and study some of their fundamental properties.

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1. INTRODUCTION

In 2000, G.B. Navalagi [1] presented the idea of semi-α-open sets in topological spaces. N.M. Ali [2] introduced new types of weakly open sets in topological spaces. M.L. Thivagar and C. Richard [3] gave nano topological space (or simply N.T.S.) on a subset M of a universe which is defined regarding lower and upper approximations of M. He studied about the weak forms of nano open sets (briefly N-O.S.), such as Nα-O.S., Nσ-O.S., and Np-O.S. The objective of this paper is to present the idea of NSₐα-O.S. and study their fundamental properties in nano topological spaces. We also present NSₐα-interior and NSₐα-closure and obtain some of its properties.

2. PRELIMINARIES

Throughout this paper, (U, τₓ(M)) (or simply U) always mean a nano topological space on which no separation axioms are expected unless generally specified. The complement of a N-O.S. is called a nano closed set (briefly N-C.S.) in (U, τₓ(M)). For a set C in a nano topological space (U, τₓ(M)), Ncl(C), Nint(C) and Cᶜ = U − C denote the nano closure of C, the nano interior of C and the nano complement of C respectively.

Definition 2.1 [3]:

A subset C of an N.T.S. (U, τₓ(M)) is said to be:
(i) A nano pre-open set (briefly Np-O.S.) if C ⊆ Nint(Ncl(C)). The complement of a Np-O.S. is called a nano pre-closed set (briefly Np-C.S.) in (U, τₓ(M)). The family of all Np-O.S. (resp. Np-C.S.) of U is denoted by NpO(U, M) (resp. NpC(U, M)).
(ii) A nano semi-open set (briefly Ns-O.S.) if C ⊆ Ncl(Nint(C)). The complement of a Ns-O.S. is called a nano semi-closed set (briefly Ns-C.S.) in (U, τₓ(M)). The family of all Ns-O.S. (resp. Ns-C.S.) of U is denoted by NsO(U, M) (resp. NsC(U, M)).
(iii) A nano α-open set (briefly Na-O.S.) if C ⊆ Nint(Ncl(Nint(C))). The complement of a Na-O.S. is called a nano α-closed set (briefly Na-C.S.) in (U, τₓ(M)). The family of all Na-O.S. (resp. Na-C.S.) of U is denoted by NaO(U, M) (resp. NaC(U, M)).
Definition 2.2 [3]:

(i) The $Np$-interior of a set $C$ of a N.T.S. $(U, \tau_R(M))$ is the union of all $Np$-O.S. contained in $C$ and is denoted by $Npint(C)$.
(ii) The $Ns$-interior of a set $C$ of a N.T.S. $(U, \tau_R(M))$ is the union of all $Ns$-O.S. contained in $C$ and is denoted by $Nsint(C)$.
(iii) The $N\alpha$-interior of a set $C$ of a N.T.S. $(U, \tau_R(M))$ is the union of all $N\alpha$-O.S. contained in $C$ and is denoted by $Naint(C)$.

Definition 2.3 [3]:

(i) The $Np$-closure of a set $C$ of a N.T.S. $(U, \tau_R(M))$ is the intersection of all $Np$-C.S. that contain $C$ and is denoted by $Npcl(C)$.
(ii) The $Ns$-closure of a set $C$ of a N.T.S. $(U, \tau_R(M))$ is the intersection of all $Ns$-C.S. that contain $C$ and is denoted by $NscI(C)$.
(iii) The $N\alpha$-closure of a set $C$ of a N.T.S. $(U, \tau_R(M))$ is the intersection of all $N\alpha$-C.S. that contain $C$ and is denoted by $Nacl(C)$.

Proposition 2.4 [3]:

In a N.T.S. $(U, \tau_R(M))$, then the following statements hold, and the equality of each statement are not true:
(i) Every $N$-O.S. (resp. $N$-C.S.) is a $N\alpha$-O.S. (resp. $N\alpha$-C.S.).
(iii) Every $N\alpha$-O.S. (resp. $N\alpha$-C.S.) is a $Ns$-O.S. (resp. $Ns$-C.S.).

Proposition 2.5 [3]:

A subset $C$ of a N.T.S. $(U, \tau_R(M))$ is a $N\alpha$-O.S. iff $C$ is a $Ns$-O.S. and $Np$-O.S.

Lemma 2.6:

(i) If $K$ is a $N$-O.S., then $Nscl(K) = Nint(Ncl(K))$.
(ii) If $C$ is a subset of a N.T.S. $(U, \tau_R(M))$, then $Nsint(Ncl(C)) = Ncl(Nint(Ncl(C)))$.

3. NANO SEMI-$\alpha$-OPEN SETS

In this section, we present and study the $NS_\alpha$-O.S. and some of its properties.

Definition 3.1:

A subset $C$ of a N.T.S. $(U, \tau_R(M))$ is called nano semi-$\alpha$-open set (briefly $NS_\alpha$-O.S.) if there exists a $N\alpha$-O.S. $P$ in $U$ such that $P \subseteq C \subseteq Ncl(P)$ or equivalently if $C \subseteq Ncl(Naint(C))$. The family of all $NS_\alpha$-O.S. of $U$ is denoted by $NS_\alpha O(U, M)$.

Definition 3.2:

The complement of $NS_\alpha$-O.S. is called a nano semi-$\alpha$-closed set (briefly $NS_\alpha$-C.S.). The family of all $NS_\alpha$-C.S. of $U$ is denoted by $NS_\alpha C(U, M)$.
Example 3.3:

Let \( U = \{p, q, r, s\} \) with \( U/R = \{\{p\}, \{r\}, \{q, s\}\} \) and \( M = \{p, q\} \).

Let \( \tau_R(M) = \{\{p\}, \{q, s\}, \{p, q, s\}, U\} \) be a N.T.S.. The N-C.S. are \( \{U, \{q, r, s\}, \{p, r\}, \{r\}\} \) and \( \phi \). The family of all \( N\alpha\)-O.S. of \( U \) is: \( N\alpha O(U, M) = \{\phi, \{p\}, \{q, s\}, \{p, q, s\}, U\} \).

The family of all \( N\alpha\)-C.S. of \( U \) is: \( N\alpha C(U, M) = \{U, \{q, r, s\}, \{p, r\}, \{r\}\} \).

The family of all \( NS_{\alpha}\)-O.S. of \( U \) is: \( NS_{\alpha}O(U, M) = N\alpha O(U, M) \cup \{\{p, r\}, \{q, r, s\}\} \).

The family of all \( NS_{\alpha}\)-C.S. of \( U \) is: \( NS_{\alpha}C(U, M) = N\alpha C(U, M) \cup \{\{q, s\}, \{p\}\} \).

Remark 3.4:

It is evident by definitions that in a N.T.S. \( (U, \tau_R(M)) \), the following hold:

(i) Every \( N\alpha\)-O.S. (resp. N-C.S.) is a \( NS_{\alpha}\)-O.S. (resp. \( NS_{\alpha}\)-C.S.).

(ii) Every \( N\alpha\)-O.S. (resp. \( N\alpha\)-C.S.) is a \( NS_{\alpha}\)-O.S. (resp. \( NS_{\alpha}\)-C.S.).

The opposite of the above remark need not be true as appeared in the following examples.

Example 3.5:

In example (3.3), the set \( \{p, r\} \) is a \( NS_{\alpha}\)-O.S. but is not \( N\alpha\)-O.S. and not \( N\alpha\)-C.S.. The set \( \{q, s\} \) is a \( NS_{\alpha}\)-C.S. but is not N-C.S. and not \( N\alpha\)-C.S..

Remark 3.6:

The concepts of \( NS_{\alpha}\)-O.S. and \( N\alpha\)-O.S. are independent, as the following example shows.

Example 3.7:

In example (3.3), then the set \( \{p, r\} \) is a \( NS_{\alpha}\)-O.S. but is not \( N\alpha\)-O.S.. The set \( \{p, r, s\} \) is a \( N\alpha\)-O.S. but is not \( NS_{\alpha}\)-O.S..

Remark 3.8:

(i) If every \( N\alpha\)-O.S. is a N-C.S. and every nowhere nano dense set is N-C.S. in any N.T.S. \( (U, \tau_R(M)) \), then every \( NS_{\alpha}\)-O.S. is a \( N\alpha\)-O.S..

(ii) If every \( N\alpha\)-O.S. is a N-C.S. in any N.T.S. \( (U, \tau_R(M)) \), then every \( NS_{\alpha}\)-O.S. is a N-C.S..

Remark 3.9:

(i) It is clear that every \( N\alpha\)-O.S. and \( N\alpha\)-O.S. of any N.T.S. \( (U, \tau_R(M)) \) is a \( NS_{\alpha}\)-O.S. (by proposition (2.5) and remark (3.4) (ii)).

(ii) A \( NS_{\alpha}\)-O.S. in any N.T.S. \( (U, \tau_R(M)) \) is a \( N\alpha\)-O.S. if every \( N\alpha\)-O.S. of \( U \) is a N-C.S. (from proposition (2.4) (iii) and remark (3.8) (ii)).

Theorem 3.10:

For any subset \( C \) of a N.T.S. \( (U, \tau_R(M)) \), \( C \in NaO(U, M) \) iff there exists a \( N\alpha\)-O.S. \( \mathcal{K} \) such that \( \mathcal{K} \subseteq C \subseteq Nint(Ncl(\mathcal{K})) \).

Proof: Let \( C \) be a \( N\alpha\)-O.S. Hence \( C \subseteq Nint(Ncl(Nint(C))) \), so let \( \mathcal{K} = Nint(C) \), we get \( Nint(C) \subseteq C \subseteq Nint(Ncl(Nint(C))) \). Then there exists a \( N\alpha\)-O.S. \( Nint(C) \) such that \( \mathcal{K} \subseteq C \subseteq Nint(Ncl(\mathcal{K})) \), where \( \mathcal{K} = Nint(C) \).
Conversely, suppose that there is a N-O.S. \( K \) such that \( K \subseteq C \subseteq \text{Nint}(\text{Ncl}(K)) \).
To prove \( C \in \text{NaO}(U, M) \).
\( K \subseteq \text{Nint}(C) \) (since \( \text{Nint}(C) \) is the largest N-O.S. contained in \( C \)).
Hence \( \text{Ncl}(K) \subseteq \text{Nint}(C) \), then \( \text{Nint}(\text{Ncl}(K)) \subseteq \text{Nint}(\text{Ncl}(\text{Nint}(C))) \).
But \( K \subseteq C \subseteq \text{Nint}(\text{Ncl}(K)) \) (by hypothesis). Then \( C \subseteq \text{Nint}(\text{Ncl}(\text{Nint}(C))) \).
Therefore, \( C \in \text{NaO}(U, M) \).

**Theorem 3.11:**

For any subset \( C \) of a N. T. S. \( (U, \tau_K(M)) \). The following properties are equivalent:

(i) \( C \in \text{NS}_O(U, M) \).

(ii) There exists a N-O.S. say \( K \) such that \( K \subseteq C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \).

(iii) \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(C)))) \).

**Proof:**

(i) \( \Rightarrow \) (ii) Let \( C \in \text{NS}_O(U, M) \). Then there exists \( P \in \text{NaO}(U, M) \), such that \( P \subseteq C \subseteq \text{Ncl}(P) \). Hence there exists \( K \) N-O.S. such that \( K \subseteq P \subseteq \text{Nint}(\text{Ncl}(K)) \) (by theorem (3.10)). Therefore, \( \text{Ncl}(K) \subseteq \text{Ncl}(P) \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \), implies that \( \text{Ncl}(P) \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \). Then \( K \subseteq P \subseteq C \subseteq \text{Ncl}(P) \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \). Therefore, \( K \subseteq C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \), for some \( K \) N-O.S.

(ii) \( \Rightarrow \) (iii) Suppose that there exists a N-O.S. \( K \) such that \( K \subseteq C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \).

We know that \( \text{Nint}(C) \subseteq C \). On the other hand, \( K \subseteq \text{Nint}(C) \) (since \( \text{Nint}(C) \) is the largest N-O.S. contained in \( C \)). Hence \( \text{Ncl}(K) \subseteq \text{Ncl}(\text{Nint}(C)) \), then \( \text{Nint}(\text{Ncl}(K)) \subseteq \text{Nint}(\text{Nint}(\text{Ncl}(K))) \), therefore \( \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(C)))) \).

But \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \) (by hypothesis).
Hence \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \subseteq \text{Ncl}(\text{Nint}(\text{Nint}(\text{Ncl}(C)))) \).
Then \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(C)))) \).

(iii) \( \Rightarrow \) (i) Let \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(C)))) \). To prove \( C \in \text{NS}_O(U, M) \).
Let \( P = \text{Nint}(C) \); we know that \( \text{Nint}(C) \subseteq C \). To prove \( C \subseteq \text{Ncl}(\text{Nint}(C)) \).
Since \( \text{Nint}(\text{Ncl}(\text{Nint}(C))) \subseteq \text{Ncl}(\text{Nint}(\text{Nint}(C))) \).
Hence, \( \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(C)))) \subseteq \text{Ncl}(\text{Nint}(\text{Nint}(\text{Ncl}(C)))) = \text{Ncl}(\text{Nint}(C)) \).
But \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(C)))) \) (by hypothesis).
Hence, \( C \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(\text{Nint}(\text{Ncl}(C))))) \subseteq \text{Ncl}(\text{Nint}(C)) \Rightarrow C \subseteq \text{Ncl}(\text{Nint}(C)) \).
Hence, there exists a N-O.S. say \( P \), such that \( P \subseteq C \subseteq \text{Ncl}(P) \).
On the other hand, \( P \) is a \( \text{NaO} \)-O.S. (since \( P \) is a N-O.S.). Hence \( C \in \text{NS}_O(U, M) \).

**Corollary 3.12:**

For any subset \( C \) of a N. T. S. \( (U, \tau_K(M)) \), the following properties are equivalent:

(i) \( C \in \text{NS}_O(U, M) \).

(ii) There exists a N-C.S. \( F \) such that \( \text{Nint}(\text{Ncl}(\text{Nint}(F))) \subseteq C \subseteq F \).

(iii) \( \text{Nint}(\text{Ncl}(\text{Nint}(\text{Nint}(C)))) \subseteq C \).

**Proof:**

(i) \( \Rightarrow \) (ii) Let \( C \in \text{NS}_O(U, M) \), then \( C^c \in \text{NS}_O(U, M) \). Hence there is \( K \) N-O.S. such that \( K \subseteq C^c \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \) (by theorem (3.11)). Hence \( \text{Ncl}(\text{Nint}(\text{Ncl}(K))) \subseteq \text{Ncl}(\text{Nint}(\text{Ncl}(K^c))) \subseteq C^c \subseteq K^c \), i.e., \( \text{Nint}(\text{Ncl}(\text{Nint}(K^c))) \subseteq C \subseteq K^c \). Let \( K^c = F \), where \( F \) is a N-C.S. in \( U \).
Then \( \text{Nint}(\text{Ncl}(\text{Nint}(F))) \subseteq C \subseteq F \), for some \( F \) N-C.S..

(ii) \( \Rightarrow \) (iii) Suppose that there exists \( F \) N-C.S. such that \( \text{Nint}(\text{Ncl}(\text{Nint}(F))) \subseteq C \subseteq F \), but \( \text{Ncl}(C) \) is the smallest N-C.S. containing \( C \). Then \( \text{Ncl}(C) \subseteq F \), and therefore:
\[
\text{Proof: Let } \{C_i\}_{i \in \Lambda} \text{ be a family of } Na-\text{O.S. of } U. \text{ To prove } U_{i \in \Lambda} C_i \text{ is a } Na-\text{O.S.,}
\]
i.e., \( U_{i \in \Lambda} C_i \subseteq N\text{int}(N\text{cl}(U_{i \in \Lambda} C_i)) \). Then \( C_i \subseteq N\text{int}(N\text{cl}(C_i)) \), \( \forall i \in \Lambda \).

Since \( U_{i \in \Lambda} N\text{int}(C_i) \subseteq N\text{int}(U_{i \in \Lambda} C_i) \) and \( U_{i \in \Lambda} N\text{cl}(C_i) \subseteq N\text{cl}(U_{i \in \Lambda} C_i) \) hold for any nano topology. We have \( U_{i \in \Lambda} C_i \subseteq U_{i \in \Lambda} N\text{int}(N\text{cl}(C_i)) \)
\[
\subseteq N\text{int}(U_{i \in \Lambda} N\text{cl}(C_i))
\]
\[
\subseteq N\text{int}(N\text{cl}(U_{i \in \Lambda} C_i)).
\]

Hence \( U_{i \in \Lambda} C_i \) is a \( Na-\text{O.S.} \).

**Theorem 3.14:**

The union of any family of \( NS_{\alpha} \text{-O.S.} \) is a \( NS_{\alpha} \text{-O.S.} \).

**Proof:** Let \( \{C_i\}_{i \in \Lambda} \) be a family of \( NS_{\alpha} \text{-O.S.} \). To prove \( U_{i \in \Lambda} C_i \) is a \( NS_{\alpha} \text{-O.S.} \).

Since \( C_i \in NS_{\alpha}O(U, M) \). Then there is a \( Na \text{-O.S. } D_i \) such that \( D_i \subseteq C_i \subseteq N\text{cl}(D_i) \), \( \forall i \in \Lambda \).

Hence \( U_{i \in \Lambda} D_i \subseteq U_{i \in \Lambda} C_i \subseteq U_{i \in \Lambda} N\text{cl}(D_i) \subseteq N\text{cl}(U_{i \in \Lambda} D_i) \).

But \( U_{i \in \Lambda} D_i \in NaO(U, M) \) (by proposition (3.13)). Hence \( U_{i \in \Lambda} C_i \in NS_{\alpha}O(U, M) \).

**Corollary 3.15:**

The intersection of any family of \( NS_{\alpha} \text{-C.S.} \) is a \( NS_{\alpha} \text{-C.S.} \).

**Proof:** This follows directly from the theorem (3.14).

**Remark 3.16:**

The intersection of any two \( NS_{\alpha} \text{-O.S.} \) is not necessary \( NS_{\alpha} \text{-O.S.} \) as in the following example.

**Example 3.17:**

In example (3.3), \( \{p, r\} \) and \( \{q, r, s\} \) are two \( NS_{\alpha} \text{-O.S.} \), but \( \{p, r\} \cap \{q, r, s\} = \{r\} \) is not \( NS_{\alpha} \text{-O.S.} \).

**Remark 3.18:**

The following diagram shows the relations among the different types of weakly \( N \text{-O.S.} \) that were studied in this section:
4. NANO SEMI-\(\alpha\)-INTERIOR AND NANO SEMI-\(\alpha\)-CLOSURE

We present \(NS_{\alpha}\)-interior and \(NS_{\alpha}\)-closure and obtain some of its properties in this section.

**Definition 4.1:**
The union of all \(NS_{\alpha}\)-O.S. in a N.T.S. \((\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))\) contained in \(\mathcal{C}\) is called \(NS_{\alpha}\)-interior of \(\mathcal{C}\) and is denoted by \(NS_{\alpha}\text{int}(\mathcal{C})\), \(NS_{\alpha}\text{int}(\mathcal{C}) = \bigcup\{\mathcal{D}: \mathcal{D} \subseteq \mathcal{C}, \mathcal{D} \text{ is a } NS_{\alpha}\text{-O.S.}\}\).

**Definition 4.2:**
The intersection of all \(NS_{\alpha}\)-C.S. in a N.T.S. \((\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))\) containing \(\mathcal{C}\) is called \(NS_{\alpha}\)-closure of \(\mathcal{C}\) and is denoted by \(NS_{\alpha}\text{cl}(\mathcal{C})\), \(NS_{\alpha}\text{cl}(\mathcal{C}) = \bigcap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D} \text{ is a } NS_{\alpha}\text{-C.S.}\}\).

**Proposition 4.3:**
Let \(\mathcal{C}\) be any set in a N.T.S. \((\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))\), the following properties are true:
(i) \(NS_{\alpha}\text{int}(\mathcal{C}) = \mathcal{C}\) iff \(\mathcal{C}\) is a \(NS_{\alpha}\)-O.S.
(ii) \(NS_{\alpha}\text{cl}(\mathcal{C}) = \mathcal{C}\) iff \(\mathcal{C}\) is a \(NS_{\alpha}\)-C.S.
(iii) \(NS_{\alpha}\text{int}(\mathcal{C})\) is the largest \(NS_{\alpha}\)-O.S. contained in \(\mathcal{C}\).
(iv) \(NS_{\alpha}\text{cl}(\mathcal{C})\) is the smallest \(NS_{\alpha}\)-C.S. containing \(\mathcal{C}\).

**Proof:** (i), (ii), (iii) and (iv) are obvious.

**Proposition 4.4:**
Let \(\mathcal{C}\) be any set in a N.T.S. \((\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))\), the following properties are true:
(i) \(NS_{\alpha}\text{int}(\mathcal{U} - \mathcal{C}) = \mathcal{U} - (NS_{\alpha}\text{cl}(\mathcal{C}))\).
(ii) \(NS_{\alpha}\text{cl}(\mathcal{U} - \mathcal{C}) = \mathcal{U} - (NS_{\alpha}\text{int}(\mathcal{C}))\).

**Proof:** (i) By definition, \(NS_{\alpha}\text{cl}(\mathcal{C}) = \bigcap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D} \text{ is a } NS_{\alpha}\text{-C.S.}\}\)
\[
\mathcal{U} - (NS_{\alpha}\text{cl}(\mathcal{C})) = \mathcal{U} - \bigcap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D} \text{ is a } NS_{\alpha}\text{-C.S.}\}\]
\[
= \bigcup\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D} \text{ is a } NS_{\alpha}\text{-C.S.}\}\]
\[
= \bigcup\{\mathcal{H}: \mathcal{H} \subseteq \mathcal{C} \text{ and } \mathcal{H} \text{ is a } NS_{\alpha}\text{-O.S.}\}\]
\[
= NS_{\alpha}\text{int}(\mathcal{U} - \mathcal{C}).
\]
Theorem 4.5:

Let $\mathcal{C}$ and $\mathcal{D}$ be two sets in a N.T.S. $(\mathcal{U}, \tau_\mathcal{M})$. The following properties hold:

(i) $NS_\alpha \text{int}(\phi) = \phi$, $NS_\alpha \text{int}(\mathcal{U}) = \mathcal{U}$.

(ii) $NS_\alpha \text{int}(\mathcal{C}) \subseteq \mathcal{C}$.

(iii) $\mathcal{C} \subseteq \mathcal{D} \Rightarrow NS_\alpha \text{int}(\mathcal{C}) \subseteq NS_\alpha \text{int}(\mathcal{D})$.

(iv) $NS_\alpha \text{int}(\mathcal{C} \cap \mathcal{D}) \subseteq NS_\alpha \text{int}(\mathcal{C}) \cap NS_\alpha \text{int}(\mathcal{D})$.

(v) $NS_\alpha \text{int}(\mathcal{C}) \cup NS_\alpha \text{int}(\mathcal{D}) \subseteq NS_\alpha \text{int}(\mathcal{C} \cup \mathcal{D})$.

(vi) $NS_\alpha \text{int}(NS_\alpha \text{int}(\mathcal{C})) = NS_\alpha \text{int}(\mathcal{C})$.

Proof: (i), (ii), (iii), (iv), (v) and (vi) are obvious.

The equality in (iv) and (v) is not true in general, as the following example shows:

Example 4.6:

Let $\mathcal{U} = \{p, q, r, s\}$ with $\mathcal{U}/\mathcal{R} = \{\{q\}, \{r\}, \{p, s\}\}$ and $\mathcal{M} = \{p, r\}$.

Let $\tau_\mathcal{M} = \{\phi, \{q\}, \{p, s\}, \{p, r, s\}, \{q\}\}$ and $\mathcal{U}$ be a N.T.S. The $N$-C.S. are $\mathcal{U}$, $\{p, q, s\}, \{q\}$ and $\phi$. The family of all $N\alpha$-O.S. of $\mathcal{U}$ is: $\mathcal{N}\alpha\mathcal{O}(\mathcal{U}, \mathcal{M}) = \{\phi, \{q\}, \{p, s\}, \{p, r, s\}, \{q\}\}$.

The family of all $NS_\alpha$-O.S. of $\mathcal{U}$ is: $NS_\alpha\mathcal{O}(\mathcal{U}, \mathcal{M}) = \mathcal{N}\alpha\mathcal{O}(\mathcal{U}, \mathcal{M}) \cup \{\{q\}, \{p, q, s\}\}$.

Let $\mathcal{C} = \{q, r\}, \mathcal{D} = \{p, q, s\}$. Then $NS_\alpha \text{int}(\mathcal{C}) = \{q, r\}, NS_\alpha \text{int}(\mathcal{D}) = \{p, q, s\}, \mathcal{C} \cap \mathcal{D} = \{q\}$, $NS_\alpha \text{int}(\mathcal{C} \cap \mathcal{D}) = \phi$ and $NS_\alpha \text{int}(\mathcal{C}) \cap NS_\alpha \text{int}(\mathcal{D}) = \{q\}$.

It is clear that $NS_\alpha \text{int}(\mathcal{C}) \cap NS_\alpha \text{int}(\mathcal{D}) \subsetneq NS_\alpha \text{int}(\mathcal{C} \cap \mathcal{D})$.

Let $\mathcal{C} = \{p, s\}, \mathcal{D} = \{q, s\}$. Then $NS_\alpha \text{int}(\mathcal{C}) = \{p, s\}, NS_\alpha \text{int}(\mathcal{D}) = \phi, \mathcal{C} \cup \mathcal{D} = \{p, q, s\}$, $NS_\alpha \text{int}(\mathcal{C} \cup \mathcal{D}) = \{p, q, s\}$ and $NS_\alpha \text{int}(\mathcal{C}) \cup NS_\alpha \text{int}(\mathcal{D}) = \{p, s\}$.

It is clear that $NS_\alpha \text{int}(\mathcal{C} \cup \mathcal{D}) \subsetneq NS_\alpha \text{int}(\mathcal{C}) \cup NS_\alpha \text{int}(\mathcal{D})$.

Theorem 4.7:

Let $\mathcal{C}$ and $\mathcal{D}$ be two sets in a N.T.S. $(\mathcal{U}, \tau_\mathcal{M})$. The following properties hold:

(i) $NS_\alpha \text{cl}(\phi) = \phi$, $NS_\alpha \text{cl}(\mathcal{U}) = \mathcal{U}$.

(ii) $\mathcal{C} \subseteq NS_\alpha \text{cl}(\mathcal{C})$.

(iii) $\mathcal{C} \subseteq \mathcal{D} \Rightarrow NS_\alpha \text{cl}(\mathcal{C}) \subseteq NS_\alpha \text{cl}(\mathcal{D})$.

(iv) $NS_\alpha \text{cl}(\mathcal{C} \cap \mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{C}) \cap NS_\alpha \text{cl}(\mathcal{D})$.

(v) $NS_\alpha \text{cl}(\mathcal{C}) \cup NS_\alpha \text{cl}(\mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{C} \cup \mathcal{D})$.

(vi) $NS_\alpha \text{cl}(NS_\alpha \text{cl}(\mathcal{C})) = NS_\alpha \text{cl}(\mathcal{C})$.

Proof: (i) and (ii) are evident.

(iii) By part (ii), $\mathcal{D} \subseteq NS_\alpha \text{cl}(\mathcal{D})$. Since $\mathcal{C} \subseteq \mathcal{D}$, we have $\mathcal{C} \subseteq NS_\alpha \text{cl}(\mathcal{D})$. But $NS_\alpha \text{cl}(\mathcal{D})$ is a $NS_\alpha$-C.S. Thus $\text{Nng}_{\alpha}\text{cl}(\mathcal{D})$ is a $NS_\alpha$-C.S. containing $\mathcal{C}$. Since $NS_\alpha \text{cl}(\mathcal{C})$ is the smallest $NS_\alpha$-C.S. containing $\mathcal{C}$, we have $NS_\alpha \text{cl}(\mathcal{C}) \subseteq NS_\alpha \text{cl}(\mathcal{D})$. Hence, $\mathcal{C} \subseteq \mathcal{D} \Rightarrow NS_\alpha \text{cl}(\mathcal{C}) \subseteq NS_\alpha \text{cl}(\mathcal{D})$.

(iv) We know that $\mathcal{C} \cap \mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{C} \cap \mathcal{D} \subseteq \mathcal{D}$. Therefore, by part (iii), $NS_\alpha \text{cl}(\mathcal{C} \cap \mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{C})$ and $S_\alpha \text{cl}(\mathcal{C} \cap \mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{D})$. Hence $NS_\alpha \text{cl}(\mathcal{C} \cap \mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{C}) \cap NS_\alpha \text{cl}(\mathcal{D})$.

(v) Since $\mathcal{C} \subseteq \mathcal{C} \cup \mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{C} \cup \mathcal{D}$, it follows from part (iii) that $NS_\alpha \text{cl}(\mathcal{C}) \subseteq NS_\alpha \text{cl}(\mathcal{C} \cup \mathcal{D})$ and $NS_\alpha \text{cl}(\mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{C} \cup \mathcal{D})$. Hence $NS_\alpha \text{cl}(\mathcal{C}) \cup NS_\alpha \text{cl}(\mathcal{D}) \subseteq NS_\alpha \text{cl}(\mathcal{C} \cup \mathcal{D})$.

(vi) Since $NS_\alpha \text{cl}(\mathcal{C})$ is a $NS_\alpha$-C.S., we have by proposition (4.3) part (ii), $S_\alpha \text{cl}(NS_\alpha \text{cl}(\mathcal{C})) = NS_\alpha \text{cl}(\mathcal{C})$. 

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The equality in (iv) and (v) is not true in general, as the following example shows:

**Example 4.8:**

In example (4.6), the family of all Na-C.S. of $\mathcal{U}$ is: $Na_{\alpha}\mathcal{C}(\mathcal{U}, \mathcal{M}) = \{U, \{p, q, s\}, \{q, r\}, \{q\}, \phi\}$. The family of all $Na_{\alpha}$C.S. of $\mathcal{U}$ is: $Na_{\alpha}\mathcal{C}(\mathcal{U}, \mathcal{M}) = Na_{\alpha}\mathcal{C}(\mathcal{U}, \mathcal{M}) \cup \{\{p, s\}, \{r\}\}$. Let $\mathcal{C} = \{p, r\}, \mathcal{D} = \{q, r\}$. Then $Na_{\alpha}\mathcal{C}(\mathcal{C}) = \mathcal{U}, Na_{\alpha}\mathcal{C}(\mathcal{D}) = \{q, r\}, \mathcal{C} \cap \mathcal{D} = \{r\}, Na_{\alpha}\mathcal{C}(\mathcal{C} \cap \mathcal{D}) = \{r\}$ and $Na_{\alpha}\mathcal{C}(\mathcal{C}) \cap Na_{\alpha}\mathcal{C}(\mathcal{D}) = \{q, r\}$.

It is clear that $Na_{\alpha}\mathcal{C}(\mathcal{C}) \cap Na_{\alpha}\mathcal{C}(\mathcal{D}) \not\subseteq Na_{\alpha}\mathcal{C}(\mathcal{C} \cap \mathcal{D})$.

Let $\mathcal{C} = \{p, s\}, \mathcal{D} = \{r\}$. Then $Na_{\alpha}\mathcal{C}(\mathcal{C}) = \{p, s\}, Na_{\alpha}\mathcal{C}(\mathcal{D}) = \{r\}, \mathcal{C} \cup \mathcal{D} = \{p, r, s\}, Na_{\alpha}\mathcal{C}(\mathcal{C} \cup \mathcal{D}) = \mathcal{U}$ and $Na_{\alpha}\mathcal{C}(\mathcal{C}) \cup Na_{\alpha}\mathcal{C}(\mathcal{D}) = \{p, r, s\}$.

It is clear that $Na_{\alpha}\mathcal{C}(\mathcal{C} \cup \mathcal{D}) \not\subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \cup Na_{\alpha}\mathcal{C}(\mathcal{D})$.

**Proposition 4.9:**

For any subset $\mathcal{C}$ of a N. T.S. ($\mathcal{U}, \tau_\alpha(\mathcal{M})$), then:

(i) $Nint(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C})$.

(ii) $Nint(Na_{\alpha}\mathcal{C}(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(iii) $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(iv) $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(v) $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(vi) $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(vii) $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(viii) $Nint(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$.

Proof: We shall prove only (ii), (iii), (iv) and (viii).

(ii) To prove $Nint(Na_{\alpha}\mathcal{C}(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

Since $Nint(\mathcal{C})$ is a N.O.S., then $Nint(\mathcal{C})$ is a $Na_{\alpha}$-O.S.

Hence $Nint(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$ (by proposition (4.3)). Therefore:

$Nint(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$ ................................................. (1)

Since $Nint(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \Rightarrow Nint(Nint(\mathcal{C})) \subseteq Nint(Na_{\alpha}\mathcal{C}(\mathcal{C})) \Rightarrow Nint(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$.

Also, $Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq \mathcal{C} \Rightarrow Nint(Na_{\alpha}\mathcal{C}(\mathcal{C})) \subseteq Nint(\mathcal{C})$. Hence:

$Nint(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$ ................................................. (2)

Therefore by (1) and (2), we get $Nint(Na_{\alpha}\mathcal{C}(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

(iii) To prove $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Nint(\mathcal{C})$.

Since $Na_{\alpha}\mathcal{C}(\mathcal{C})$ is $Na_{\alpha}$-O.S., therefore $Na_{\alpha}\mathcal{C}(\mathcal{C})$ is $Na_{\alpha}$-O.S. Therefore by proposition (4.3):

$Na_{\alpha}\mathcal{C}(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$ ................................................. (1)

Now, to prove $Na_{\alpha}\mathcal{C}(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$.

Since $Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C}) \Rightarrow Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) \subseteq Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) \Rightarrow Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$.

Also, $Na_{\alpha}\mathcal{C}(\mathcal{C}) \subseteq \mathcal{C} \Rightarrow Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) \subseteq Na_{\alpha}\mathcal{C}(\mathcal{C})$. Hence:

$Na_{\alpha}\mathcal{C}(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$ ................................................. (2)

Therefore by (1) and (2), we get $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(\mathcal{C})$.

(iv) To prove $Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C})) = Na_{\alpha}\mathcal{C}(\mathcal{C})$.

We know that $Na_{\alpha}\mathcal{C}(\mathcal{C})$ is a $Na_{\alpha}$-C.S., so it is $Na_{\alpha}$-C.S. Hence by proposition (4.3), we have:

$Na_{\alpha}\mathcal{C}(\mathcal{C}) = Na_{\alpha}\mathcal{C}(Nint(\mathcal{C}))$ ................................................. (1)
To prove $Ncl(\mathcal{E}) = Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{E}))$. Since $\text{NS}_{\alpha}\text{cl}(\mathcal{E}) \subseteq Ncl(\mathcal{E})$ (by part (i)).

Then $Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{E})) \subseteq Ncl(Ncl(\mathcal{E})) = Ncl(\mathcal{E}) \implies Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{E})) \subseteq Ncl(\mathcal{E})$.

Since $\mathcal{C} \subseteq \text{NS}_{\alpha}\text{cl}(\mathcal{E}) \subseteq Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{E}))$, then $\mathcal{C} \subseteq Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{E}))$. Hence $Ncl(\mathcal{C}) \subseteq Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C})) = Ncl(\mathcal{C}) \implies Ncl(\mathcal{C}) \subseteq Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C}))$ and therefore:

$Ncl(\mathcal{C}) = Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C}))$ .................................................................(2)

Now, by (1) and (2), we get that $Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C})) = \text{NS}_{\alpha}\text{cl}(Ncl(\mathcal{C}))$.

Hence $Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C})) = \text{NS}_{\alpha}\text{cl}(Ncl(\mathcal{C})) = Ncl(\mathcal{C})$.

(vii) To prove $\text{NS}_{\alpha}\text{int}(C) = C \cap Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{int}(\mathcal{C}))))$.

Since $\text{NS}_{\alpha}\text{int}(\mathcal{C}) \in \text{NS}_{\alpha}O(\mathcal{U}, \mathcal{M}) \implies \text{NS}_{\alpha}\text{int}(\mathcal{C}) \subseteq Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{int}(\mathcal{C}))))$

$\implies Ncl(Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{int}(\mathcal{C}))))))$ (by proposition (4.3)).

Also, $\text{NS}_{\alpha}\text{int}(\mathcal{C})$ contained in $\mathcal{C}$. Then $\mathcal{C} \cap Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{int}(\mathcal{C}))))$ contained in $\text{NS}_{\alpha}\text{int}(\mathcal{C})$ (since $\text{NS}_{\alpha}\text{int}(\mathcal{C})$ is the largest $\text{NS}_{\alpha}$-O.S. contained in $\mathcal{C}$). Hence:

$\mathcal{C} \cap Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{int}(\mathcal{C})))) \subseteq \text{NS}_{\alpha}\text{int}(\mathcal{C})$ .............................................(2)

By (1) and (2), $\text{NS}_{\alpha}\text{int}(\mathcal{C}) = C \cap Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{int}(\mathcal{C}))))$.

(viii) To prove that $Nint(\mathcal{C}) \subseteq \text{NS}_{\alpha}\text{int}(\text{NS}_{\alpha}\text{cl}(\mathcal{C}))$. Since $\text{NS}_{\alpha}\text{cl}(\mathcal{C})$ is a $\text{NS}_{\alpha}$-C.S., therefore $Nint(Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C}))) \subseteq \text{NS}_{\alpha}\text{cl}(\mathcal{C})$ (by corollary (3.12)).

Hence $Nint(\mathcal{C}) \subseteq Ncl(Nint(Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{C})))) \subseteq \text{NS}_{\alpha}\text{cl}(\mathcal{C})$ (by part (iv)).

Therefore, $\text{NS}_{\alpha}\text{int}(Nint(Ncl(\mathcal{C}))) \subseteq \text{NS}_{\alpha}\text{int}(\text{NS}_{\alpha}\text{cl}(\mathcal{C}))$ implies

$Nint(Ncl(\mathcal{C})) \subseteq \text{NS}_{\alpha}\text{int}(\text{NS}_{\alpha}\text{cl}(\mathcal{C}))$ (by part (ii)).

**Theorem 4.10:**

For any subset $\mathcal{C}$ of a N. T. S. ($\mathcal{U}, \tau_{\mathcal{K}}(\mathcal{M})$). The following properties are equivalent:

(i) $\mathcal{C} \in \text{NS}_{\alpha}O(\mathcal{U}, \mathcal{M})$.

(ii) $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(\text{NS}_{\alpha}\text{cl}(\mathcal{K})))$, for some N-O.S. $\mathcal{K}$.

(iii) $\mathcal{K} \subseteq \mathcal{C} \subseteq N\text{nsint}(\text{NS}_{\alpha}\text{cl}(\mathcal{K}))$, for some N-O.S. $\mathcal{K}$.

(iv) $\mathcal{C} \subseteq N\text{nsint}(Nint(\text{NS}_{\alpha}\text{cl}(\mathcal{K})))$.

**Proof:**

(i) $\implies$ (ii) Let $\mathcal{C} \in \text{NS}_{\alpha}O(\mathcal{U}, \mathcal{M})$, then $\mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{C})))$ and $Nint(\mathcal{C}) \subseteq \mathcal{C}$. Hence $\mathcal{K} \subseteq Ncl(Nint(\text{NS}_{\alpha}\text{cl}(\mathcal{K})))$, where $\mathcal{K} = Nint(\mathcal{C})$.

(ii) $\implies$ (iii) Suppose $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(\text{NS}_{\alpha}\text{cl}(\mathcal{K})))$, for some N-O.S. $\mathcal{K}$.

But $N\text{nsint}(\text{NS}_{\alpha}\text{cl}(\mathcal{K})) = Ncl(Nint(\text{NS}_{\alpha}\text{cl}(\mathcal{K})))$ (by lemma (2.6)).

Then $\mathcal{K} \subseteq \mathcal{C} \subseteq N\text{nsint}(Ncl(\text{NS}_{\alpha}\text{cl}(\mathcal{K})))$, for some N-O.S. $\mathcal{K}$.

(iii) $\implies$ (iv) Suppose that $\mathcal{K} \subseteq \mathcal{C} \subseteq N\text{nsint}(Ncl(\mathcal{K}))$, for some N-O.S. $\mathcal{K}$.

Since $\mathcal{K}$ is a N-O.S. contained in $\mathcal{C}$. Then $\mathcal{K} \subseteq Nint(\mathcal{C}) \implies Ncl(\mathcal{K}) \subseteq Ncl(Nint(\mathcal{C}))$ and therefore:

$N\text{nsint}(Ncl(\mathcal{K})) \subseteq N\text{nsint}(Ncl(Nint(\mathcal{C})))$. But $\mathcal{C} \subseteq N\text{nsint}(Ncl(\mathcal{K}))$ (by hypothesis),
then \( \mathcal{C} \subseteq Nsint(Ncl(Nint(\mathcal{C}))) \).

(iv) \( \Rightarrow \) (i) Let \( \mathcal{C} \subseteq Nsint(Ncl(Nint(\mathcal{C}))) \).

But \( Nsint(Ncl(Nint(\mathcal{C}))) = Ncl(Nint(Ncl(Nint(\mathcal{C})))) \) (by lemma (2.6)).

Hence \( \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C})))) \Rightarrow \mathcal{C} \in NS_\alpha O(\mathcal{U} , \mathcal{M}) \).

**Corollary 4.11:**

For any subset \( \mathcal{D} \) of a N. T. S. \( (\mathcal{U} , \tau_\alpha(\mathcal{M})) \), the following properties are equivalent:

(i) \( \mathcal{D} \in NS_\alpha C(\mathcal{U} , \mathcal{M}) \).

(ii) \( Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{D} \subseteq \mathcal{F} \), for some \( \mathcal{F} \) N-C.S.

(iii) \( Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \subseteq \mathcal{F} \), for some \( \mathcal{F} \) N-C.S.

(iv) \( Nscl(Nint(Ncl(\mathcal{D}))) \subseteq \mathcal{D} \).

**Proof:**

(i) \( \Rightarrow \) (ii) Let \( \mathcal{D} \in NS_\alpha C(\mathcal{U} , \mathcal{M}) \Rightarrow Nint(Ncl(Nint(Ncl(\mathcal{D})))) \subseteq \mathcal{D} \) (by corollary (3.12)) and \( \mathcal{D} \subseteq Ncl(\mathcal{D}) \). Hence we get \( Nint(Ncl(Nint(Ncl(\mathcal{D})))) \subseteq \mathcal{D} \subseteq Ncl(\mathcal{D}) \).

Therefore \( Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{D} \subseteq \mathcal{F} \), where \( \mathcal{F} = Ncl(\mathcal{D}) \).

(ii) \( \Rightarrow \) (iii) Let \( Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{D} \subseteq \mathcal{F} \), for some \( \mathcal{F} \) N-C.S.

But \( Nint(Ncl(Nint(\mathcal{F}))) = Nscl(Nint(\mathcal{F})) \) (by lemma (2.6)).

Hence \( Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \subseteq \mathcal{F} \), for some \( \mathcal{F} \) N-C.S.

(iii) \( \Rightarrow \) (iv) Let \( Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \subseteq \mathcal{F} \), for some \( \mathcal{F} \) N-C.S.

Since \( \mathcal{D} \subseteq \mathcal{F} \) (by hypothesis), hence \( Ncl(\mathcal{D}) \subseteq \mathcal{F} \Rightarrow Nint(Ncl(\mathcal{D}) \subseteq Nint(\mathcal{F}) \Rightarrow Nscl(Nint(\mathcal{D}))) \subseteq Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \Rightarrow Nscl(Nint(Ncl(\mathcal{D}))) \subseteq \mathcal{D} \).

(iv) \( \Rightarrow \) (i) Let \( nscl(Nint(Ncl(\mathcal{D}))) \subseteq \mathcal{D} \).

But \( Nscl(Nint(Ncl(\mathcal{D}))) = Nint(Ncl(Nint(Ncl(\mathcal{D})))) \) (by lemma (2.6)).

Hence \( Nint(Ncl(Nint(Ncl(\mathcal{D})))) \subseteq \mathcal{D} \Rightarrow \mathcal{D} \in NS_\alpha C(\mathcal{U} , \mathcal{M}) \).

5. CONCLUSION

The class of \( NS_\alpha \)-O.S. defined using \( N\alpha \)-O. S. forms a nano topology and lay between the class of \( N \)-O.S. and the class of \( Ns \)-O.S. The \( NS_\alpha \)-O.S. can be used to derive a new decomposition of nano continuity, nano compactness, and nano connectedness.

REFERENCES