NON-ININVARIANT HYPERSURFACES OF A NEARLY SASAKIAN MANIFOLD WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. The present paper focuses on the study of non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection equipped with structure. Firstly, some properties of this structure are obtained. Further, the second fundamental forms of non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection has been traced under the condition when is parallel. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection with structure of nearly Sasakian manifold to be totally geodesic.

Keywords: Nearly Sasakian, Semi-symmetric non-metric connection, Totally umbilical, Totally geodesic.

1. INTRODUCTION

Goldberg and Yano [14], in 1970’s studied the notion of a non–invariant hypersurface of an almost contact manifold such that transform of a tangent vector of hypersurface by the (1,1) structure tensor field defining the almost contact structure is never tangent to the hypersurface. Yano studied induced structures on submanifolds [2]. Yano et al [3–5, 7], introduced (f, g, u, v, λ)-structure and termed it as a non–invariant hypersurface of an almost contact metric manifold and studied their properties. A hypersurface of an almost contact manifold always admits a (f, g, u, v, λ)-structure was studied by Blair and Yano in [1] and [6] respectively. Prasad [12] studied the non–invariant hypersurfaces of trans-Sasakian manifold. In 2011, Prasad and Kishore [13] studied non-invariant hypersurfaces of nearly Sasakian manifold. In the present paper, we study non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection.

2. PRELIMINARIES

Let \( \bar{M} \) be an almost contact metric manifold with the almost contact metric structure \((\emptyset, \xi, \eta, g)\), where a tensor \( \emptyset \) of type (1,1), a vector field \( \xi \), called structure vector field and \( \eta \), the dual 1-form of \( \xi \) and \( g \) is a compatible Reimannian metric such that

\[
\emptyset^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \emptyset(\xi) = 0, \quad \eta\emptyset = 0, \quad (1)
\]

\[
g(\emptyset X, \emptyset Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2)
\]

\[
g(X, \emptyset Y) = -g(\phi X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \quad (3)
\]

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for all \( X, Y \in T\bar{M} \).

An almost contact metric manifold is a nearly Sasakian manifold if
\[
(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y. \tag{4}
\]
Now, we define a semi-symmetric non-metric connection by\[10], \[11]\n\[
\nabla_X g(Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(\phi X, Y). \tag{5}
\]
Using (5) and (4), we have
\[
(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y - \eta(Y)\phi X - \eta(X)\phi Y. \tag{6}
\]
An almost contact manifold \( \bar{M} \) satisfying (6) is called non-invariant hypersurfaces of a nearly Sasakian manifold with semi-symmetric non-metric connection.

For a non-invariant hypersurfaces of a nearly Sasakian manifold with semi-symmetric non-metric connection, we have
\[
\nabla_X \xi = X - \phi X - \eta(X)\xi - \phi (\nabla_\xi \phi)X. \tag{7}
\]
A hypersurface of an almost contact metric manifold \( \bar{M} \) \((\emptyset, \xi, \eta, g)\) is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of (1,1) tensor field \( \phi \) defining the contact structure is never tangent to the hypersurface. Let \( X \) be a tangent vector on a non-invariant hypersurface of an almost contact metric manifold \( \bar{M} \), then \( \phi X \) is never tangent to the hypersurface.

Let \( M \) be a non-invariant hypersurface of an almost contact metric manifold. Now if we define the following:
\[
\phi X = fX + u(X)\tilde{N}, \tag{8}
\]
\[
\phi \tilde{N} = -U, \tag{9}
\]
\[
\xi = V + \lambda \tilde{N}, \quad \lambda = \eta(\tilde{N}), \tag{10}
\]
\[
\eta(X) = v(X), \tag{11}
\]
where \( f \) is a (1,1) tensor field, \( u \) and \( v \) are 1-forms, \( \tilde{N} \) is a unit normal to the hypersurface, \( X \in TM \) and \( u(X) \neq 0 \); then we get an induced a \((f, g, u, v, \lambda)\)-structure \( [3] \) on \( M \) satisfying the conditions:
\[
f^2 = -I + u \otimes U + v \otimes V, \tag{12}
\]
\[
fU = -\lambda V, \quad fV = \lambda U, \tag{13}
\]
\[
u \circ f = \lambda v, \quad v \circ f = -\lambda u, \tag{14}
\]
\[
v(V) = 1 - \lambda^2, \quad v(U) = u(V) = 0, u(U) = 1 - \lambda^2, \tag{15}
\]
\[
g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y), \tag{16}
\]
\[
g(X, fY) = -g(fX, Y), \quad g(X, U) = u(X), \quad g(X, V) = v(X), \tag{17}
\]
for all \( X, Y \in TM \), where \( \lambda = \eta(\tilde{N}) \).

The Gauss and Weingarten formulae for a non-invariant hypersurfaces of a nearly Sasakian manifold with semi-symmetric non-metric connection is given by
\[
\nabla_X Y = \nabla_X Y + \sigma(X, Y)\tilde{N}, \tag{18}
\]
\[
\nabla_X \tilde{N} = -A_R X + \lambda X \tag{19}
\]
for all \( X, Y \in TM \), where \( \nabla \) and \( \nabla \) are the Riemannian and induced Riemannian connections on \( \bar{M} \) and \( M \) respectively and \( \tilde{N} \) is the unit normal vector in the normal bundle \( T^\perp M \). In this formula \( \sigma \) is the second fundamental form on \( M \) related to \( A_R \) by
\[
\sigma(X, Y) = g(A_R X, Y) \tag{20}
\]
for all \( X, Y \in TM \).
3. NON-ININVARIANT HYPERSURFACES

Lemma 3.1 If $M$ be a non-invariant hypersurface with $(f, g, u, v, \lambda)$-structure of a nearly Sasakian manifold $\tilde{M}$ with semi-symmetric non-metric connection, then

\[
(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = ((\nabla_X f)Y + (\nabla_Y f)X + \sigma(X, fY) + \sigma(Y, fX))\tilde{N} + (\nabla_X f)Y + (\nabla_Y f)X + 2\sigma(X, Y)U - u(X)A_R Y - u(Y)A_N Y + u(Y)\lambda X,
\]

(21)

\[
(\tilde{\nabla}_X \eta)Y + (\tilde{\nabla}_Y \eta)X = (\nabla_X u)Y + (\nabla_Y u)X - 2\lambda \sigma(X, Y),
\]

(22)

\[
\tilde{\nabla}_X \xi = \nabla_X V - \lambda A_R X + \lambda^2 X + (\sigma(X, V) + \lambda X)\tilde{N}
\]

(23)

for all $X, Y \in TM$.

Proof. By covariant differentiation, we know that

\[
(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)
\]

(24)

Similarly,

\[
(\tilde{\nabla}_Y \phi)X = (\nabla_Y f)X - u(X)A_R Y + \sigma(X, Y)U + u(Y)\lambda X
\]

+ ((\nabla_X u)Y + \sigma(X, fY))\tilde{N}.

(25)

From (24) and (25), we have

\[
(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = (\nabla_X f)Y + (\nabla_Y f)X - u(Y)A_R X - u(X)A_N Y + 2\sigma(X, Y) + u(Y)\lambda X + u(X)\lambda Y +
\]

\[
((\nabla_X u)Y + (\nabla_Y u)X + \sigma(Y, fX) + \sigma(X, fY))\tilde{N}.
\]

(26)

Using Gauss formula, we get

\[
(\tilde{\nabla}_X \eta)Y = (\nabla_X u)Y - \eta(\nabla_X Y).
\]

(27)

Similarly,

\[
(\tilde{\nabla}_Y \eta)X = (\nabla_Y u)X - \lambda \sigma(X, Y).
\]

(28)

Adding (26) and (27), we get

\[
(\tilde{\nabla}_X \eta)Y + (\tilde{\nabla}_Y \eta)X = (\nabla_X u)Y + (\nabla_Y u)X - 2\lambda \sigma(X, Y).
\]

(29)

Further consider,

\[
\tilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi)\tilde{N}, \quad \text{then}
\]

\[
\tilde{\nabla}_X \xi = \nabla_X V + \lambda \nabla_X \tilde{N} + (\nabla_X \lambda)\tilde{N} + \sigma(X, V)\tilde{N}
\]

\[
\tilde{\nabla}_X \xi = (\nabla_X V - \lambda A_R X + \lambda^2 X) + (\sigma(X, V) + \lambda X)\tilde{N}.
\]

Theorem 3.2. If $M$ be a non-invariant hypersurface with $(f, g, u, v, \lambda)$-structure of a nearly Sasakian manifold $\tilde{M}$, with semi-symmetric non-metric connection, then

\[
\sigma(X, \xi)U = -fX + f^2 X - u(X)U + f^2((\nabla_X \phi)X) - u((\nabla_X \phi)X)U + f(\nabla_X \xi),
\]

(28)

\[
u(\nabla_X \xi) = u(X) - u(fX) - u(f((\nabla_X \phi)X))
\]

(29)

for all $X, Y \in TM$.

Proof. Let us consider

\[
(\tilde{\nabla}_X \phi)\xi = \tilde{\nabla}_X \phi \xi - \phi(\tilde{\nabla}_X \xi)
\]

(30)

\[
(\tilde{\nabla}_X \phi)\xi = -\phi(X - fX - u(X)\tilde{N} - \eta(X)\xi - f((\nabla_X \phi)X) - u((\nabla_X \phi)X)\tilde{N})
\]

(31)
Putting (32) and (33) in (37), we get
\[
(\overline{\nabla}_X \phi)_\xi = -(fX + u(\xi)\bar{N}) + f^2X + u(fX)\bar{N} - u(X)U \\
+ f^2((\overline{\nabla}_\xi \phi)_X) + u((\overline{\nabla}_\xi \phi)_X)U - u(f((\overline{\nabla}_\xi \phi)_X))\bar{N}.
\] (30)

Since, we know the relation
\[
(\overline{\nabla}_X \phi)_\xi = -\phi(\overline{\nabla}_X \xi) + \sigma(X, \xi)U.
\] (31)
Comparing (30) and (31) and equating tangential and normal part, we get the desired results.
Hence theorem is proved.

**Theorem 3.3.** If M be a non-invariant hypersurface with \((f, g, u, v, \lambda)\)-structure of a nearly Sasakian manifold \(\bar{M}\), with semi-symmetric non-metric connection, then
\[
(\nabla_X f)Y + (\nabla_Y f)X = 2g(X, Y)V - v(X)Y - v(Y)X - v(X)\phi Y \\
- v(Y)\phi X - 2\sigma(X, Y)U + u(Y)A_R X + u(X)A_R Y - u(Y)\lambda X - u(X)\lambda Y,
\] (32)
and
\[
(\nabla_X u)Y + (\nabla_Y u)X = 2\lambda g(X, Y) - \sigma(X, fY) - \sigma(fX, Y)
\] (33)
for all \(X, Y \in TM\).

**Proof.** In view of (21) and (6), we have
\[
((\nabla_X u)Y + (\nabla_Y u)X + \sigma(X, fY)\bar{N} + \sigma(Y, fX))\bar{N} + (\nabla_X f)Y + (\nabla_Y f)X + 2\sigma(X, Y)U - u(X)A_R Y - u(Y)A_R X + u(X)\lambda Y + u(Y)\lambda X \\
= 2g(X, Y)V + 2\lambda g(X, Y)\bar{N} - v(X)Y - v(Y)X - v(Y)\phi X - v(X)\phi Y.
\]
Equating tangential and normal components of above equations, we can obtain (32) and (33) respectively.
Hence theorem is proved.

**Theorem 3.4.** If M be a non-invariant hypersurface with \((f, g, u, v, \lambda)\)-structure of a nearly Sasakian manifold \(\bar{M}\), with semi-symmetric non-metric connection, then
\[
(\overline{\nabla}_X \phi)_Y + (\overline{\nabla}_Y \phi)_X = 2\lambda g(X, Y)\bar{N} + 2g(X, Y)V - v(X)Y - v(Y)X \\
- v(X)\phi Y - v(Y)\phi X
\] (34)
for all \(X, Y \in TM\).

**Proof.** Consider,
\[
(\overline{\nabla}_X \phi)_Y = \overline{\nabla}_X \phi_Y - \phi(\overline{\nabla}_X Y)
\]
\[
(\overline{\nabla}_Y \phi)_X = \overline{\nabla}_Y \phi_X + \sigma(X, fY)\bar{N} + \overline{\nabla}_X u(Y)\bar{N} - f(\overline{\nabla}_Y Y) - u(\overline{\nabla}_X Y)\bar{N} - \sigma(X, Y)\phi \bar{N}
\]
\[
(\overline{\nabla}_X \phi)_Y = (\nabla_X f)Y + ((\nabla_Y u)X + \sigma(Y, fX))\bar{N} - u(Y)A_R X + u(Y)\lambda X + \sigma(X, Y)U.
\] (35)
Similarly,
\[
(\overline{\nabla}_Y \phi)_X = (\nabla_Y f)X + ((\nabla_Y u)X + \sigma(Y, fX))\bar{N} - u(X)A_R Y + u(X)\lambda Y + \sigma(X, Y)U.
\] (36)
Adding (35) and (36), we have
\[
(\overline{\nabla}_X \phi)_Y + (\overline{\nabla}_Y \phi)_X = ((\nabla_X u)Y + (\nabla_Y u)X + \sigma(Y, fX) + \sigma(X, fY))\bar{N} + (\nabla_X f)Y + (\nabla_Y f)X - u(Y)A_R X - u(X)A_R Y + u(X)\lambda Y + u(Y)\lambda X + 2\sigma(X, Y)U.
\] (37)
Putting (32) and (33) in (37), we get
\[
(\overline{\nabla}_X \phi)_Y + (\overline{\nabla}_Y \phi)_X = 2\lambda g(X, Y)\bar{N} + 2g(X, Y)V - v(X)Y - v(Y)X - v(X)\phi Y - v(Y)\phi X.
\]
Hence theorem is proved.

**Theorem 3.5.** If M be a totally umbilical non-invariant hypersurface with \((f, g, u, v, \lambda)\)-structure of a nearly Sasakian manifold \(\bar{M}\), with semi-symmetric non-metric connection. Then, it is totally geodesic if and only if
\[
u((\overline{\nabla}_\xi \phi)_X) + \lambda v(X) + u(X) + \lambda X = 0.
\] (38)
In particular, if nearly Sasakian manifold with semi-symmetric non-metric connection admits a contact structure then (38) can be expressed as
\[ u(X) + \lambda v(X) + \lambda X = 0 \] (39)
for all \( X, Y \in TM \).

**Proof.** From (10), we have
\[
\nabla_X \xi = \nabla_X (V + \lambda \hat{N})
= \nabla_X V + (\nabla_X \lambda) N + \lambda (\nabla_X N).
\]
Using (18) & (19), we get
\[
\nabla_X \xi = (\nabla_X V + \lambda^2 X - A_R X) + (X \lambda + \sigma(X, V)) N.
\] (40)
From (7) and (40)
\[
\nabla_X V - \lambda A_R X + \lambda^2 X + (\sigma(X, V) + \lambda X) \nabla_X \xi = X - fX - u(X) \hat{N} - v(X)(V + \lambda \hat{N}) - f((\nabla_X \xi)X) - u((\nabla_X \xi)X) \hat{N}.
\]
Equating normal part, we have
\[
\sigma(X, V) = -u((\nabla_X \xi)X) - \lambda v(X) - u(X) - \lambda X.
\] (41)
Now, if \( M \) is totally umbilical, then \( A_R = \zeta I \), where \( \zeta \) is Kahlerian metric and (20) reduces to
\[
\sigma(X, Y) = g(A_R X, Y).
\]
Therefore,
\[
\sigma(X, Y) = \zeta g(X, Y),
\]
\[
\sigma(X, \xi) = \zeta g(X, \xi) = \zeta \eta(X),
\]
\[
\sigma(X, \xi) = \zeta v(X).
\] (42)
So, (41) reduces as
\[
\zeta v(X) = u((\nabla_X \xi)X) - \lambda v(X) - u(X) - \lambda X.
\] (43)
If \( M \) is totally umbilical, that is \( \zeta = 0 \), then above becomes
\[
u((\nabla_X \xi)X) - \lambda v(X) - u(X) - \lambda X = 0.
\] (44)
Now, if nearly Sasakian manifold with semi-symmetric non-metric connection is equipped with contact structure then above can be written as
\[
\lambda v(X) + u(X) + \lambda X = 0.
\]
Hence theorem is proved.

**Theorem 3.6.** If \( M \) be a non-invariant hypersurface with \( (f, g, u, v, \lambda) \)- structure of a nearly Sasakian manifold \( \tilde{M} \), with semi-symmetric non-metric connection. If \( f \) is parallel, then we have
\[
\sigma(X, Y) = \frac{\mu - 3 \lambda (1 - \lambda^2)}{(1 - \lambda^2)^2} u(X)u(Y) - \frac{2}{1 - \lambda^2} (v(X)u(Y) + v(Y)u(X)),
\] (45)
where \( \mu = \sigma(U, U) = g(A_R U, U) \).
Also, \( M \) is totally geodesic if and only if
\[
u((\nabla_X \xi)X) + \lambda v(X) - u(X) + \lambda X = 0.
\] (46)

**Proof.** Since \( f \) is parallel then equation (32) reduces to
\[
2 \sigma(X, Y)U = 2g(X, Y)V + u(X)A_R Y + u(Y)A_R X - v(X)Y - v(Y)X - v(X)\phi Y - v(Y)\phi X - u(X)\lambda Y - u(Y)\lambda X.
\]
Applying \( u \) both sides, we get
\[
2 \sigma(X, Y)u(U) = 2g(X, Y)u(V) + u(X)u(A_R Y) + u(Y)u(A_R X) - v(X)u(Y) - v(Y)u(X) - v(X)u(\phi Y) - v(Y)u(\phi X) - u(X)u(\lambda Y) - u(Y)u(\lambda X) + 2(1 - \lambda^2) \sigma(X, Y)u(A_R Y) + u(Y)u(A_R X) - v(X)u(Y) - v(Y)u(X).
\] (47)
In view of (47), we have
\[ 2(1 - \lambda^2)\sigma(X, U) = u(X)u(A_\xi U) + u(U)u(A_\xi X) - v(X)v(U) \]
\[ -v(U)v(X) - 2\lambda u(U)v(X). \]

As,
\[ h(X, Y) = g(A_\xi X, Y) \]
\[ h(X, U) = g(A_\xi X, U) = u(A_\xi X). \]

So, above equation becomes
\[ u(A_\xi X) = \left( \frac{\mu}{1 - \lambda^2} - 2\lambda \right) u(X) - v(X), \] (48)

where \( \mu = \sigma(U, U) \).

Following in similar way, we get
\[ u(A_\xi Y) = \left( \frac{\mu}{1 - \lambda^2} - 2\lambda \right) u(Y) - v(Y). \] (49)

In view of equations (47), (48) and (49), we get
\[ \sigma(X, Y) = \frac{\mu - 3\lambda(1 - \lambda^2)}{(1 - \lambda^2)^2} u(X)u(Y) - \frac{2}{1 - \lambda^2} (v(X)v(Y) + v(Y)v(X)). \]

Next, from (41) and (45), we have
\[ u(\nabla_\xi \emptyset X) + \lambda v(X) - u(X) + \lambda X = 0. \]

Further, if nearly Sasakian manifold with semi-symmetric non-metric connection posses contact structure then
\[ \lambda v(X) - u(X) + \lambda X = 0. \]

Hence theorem is proved.

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