MODULES THAT HAVE A WEAK $\delta$-SUPPLEMENT IN EVERY COFINITE EXTENSION

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Abstract. In this paper, we study on modules that have a weak (ample) $\delta$-supplement in every extension which are adapted Zöschinger’s modules with the properties ($E$) and ($EE$). It is shown that: (1) Direct summands of modules with the property $\delta$-($CWE$) have the property $\delta$-($CWE$); (2) For a module $M$, if every submodule of $M$ has the property $\delta$-($CWE$) then so does $M$; (3) For a ring $R$, $R$ is $\delta$-semilocal iff every $R$-module has the property $\delta$-($CWE$); (4) Every factor module of a finitely generated module that has the property $\delta$-($CWE$) also has the property $\delta$-($CWE$) under a special condition; (5) Let $M$ be a module and $L$ be a submodule of $M$ such that $L \ll_{\delta} M$. If the factor module $M/L$ has the property $\delta$-($CWE$), then so does $M$; (6) On a semisimple module the concepts of modules that have the property $\delta$-($CE$) and $\delta$-($CWE$) coincide with each other.

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1. INTRODUCTION

Throughout this paper, we assume that all rings are associative with identity and all modules are unital left modules. By $X \leq M$, we mean $X$ is a submodule of $M$ or $M$ is an extension of $X$. A submodule $K \leq M$ is called small in $M$ (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule $T$ of $M$. Dually, a submodule $L \leq M$ is called essential in $M$ (denoted by $L \subseteq M$) if $L \cap X \neq 0$ for every nonzero submodule $X$ of $M$. Let $U$ and $V$ be submodules of $M$. $V$ is called a supplement of $U$ in $M$ if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V = 0$ [13]. A submodule $S$ of a module $M$ has ample supplements in $M$ if every submodule $T$ such that $M = S + T$ containing submodule has a supplement in $M$ and it is called amply supplemented if every submodule has ample supplements in $M$. If $M = U + V$ and $U \cap V \ll M$, then $V$ is called a weak supplement of $U$ in $M$, and $M$ is a weakly supplemented module if every submodule of $M$ has a weak supplement in $M$.

Recall that a submodule $N$ of a module $M$ is said to be $\delta$-small in $M$, written $N \ll_{\delta} M$, provided $M \neq N + X$ for any proper submodule $X$ of $M$ with $M/X$ singular [14]. Let $L$ be a submodule of a module $M$. A submodule $K$ of $M$ is called a $\delta$-supplement of $L$ in $M$ provided $M = L + K$ and $M \neq L + X$ for any proper submodule $X$ of $K$ with $K/X$ singular, equivalently, $M = L + K$ and $L \cap K \ll_{\delta} K$. The module $M$ is called $\delta$-supplemented if every submodule of $M$ has a $\delta$-supplement in $M$ [4]. On the other hand the submodule $N$ is said to have ample $\delta$-supplement in $M$ if every submodule $L$ of $M$ with $M = N + L$ contains a

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\(\delta\)-supplement of \(N\) in \(M\). The module \(M\) is called *amply \(\delta\)-supplemented* if every submodule of \(M\) has ample \(\delta\)-supplements in \(M\) [11]. Let \(P\) be the class of all singular simple modules and \(M\) be a module. Then \(\delta(M) = \cap \{N \leq M \mid M/N \in P\} = \sum \{N \leq M \mid N \ll \delta M\} \).

Zöschinger generalized injective modules to modules with the property \((E)\). He said that a module \(M\) has the property \((E)\) if \(M\) has a supplement in every extension. He also said that a module \(M\) has the property \((EE)\) if \(M\) has ample supplements in every extension [15]. In [4], a submodule \(M\) of a module \(N\) is called cofinite if the factor module \(N/M\) is finitely generated. Adapting Zöschinger’s module with the properties \((E)\) and \((EE)\), Çalışıcı and Türkmen say that a module \(M\) has the property \((CE)\) \(((CEE)\)) if \(M\) has a supplement (ample supplements) in every cofinite extension. Following this, in [9] the authors introduced modules with the properties \((CWE)\) and \((CWEE)\).

Generalizing Zöschinger’s module with the properties \((E)\) and \((EE)\) in [7] the authors introduced the concepts of modules with the properties \(\delta-(CE)\) and \(\delta-(CEE)\) and investigate basic properties of them. In conclusion, we show that if every submodule of a module \(M\) has the property \(\delta-(CWE)\), then \(M\) has the property \(\delta-(CWEE)\). Moreover, if \(M\) has the property \(\delta-(CWE)\), then every direct summand of \(M\) has the property \(\delta-(CWE)\). We prove that over a left hereditary ring every factor module of a finitely generated module that has the property \(\delta-(CWE)\) also has the property \(\delta-(CWE)\). In addition, we give a characterization for \(\delta\)-semilocal rings by using the property \(\delta-(CWE)\) and over a \(\delta-V\)-ring the concepts of modules with the properties \(\delta-(CWE)\) and \(\delta-(CE)\) coincide.

**2. MAIN RESULTS**

**Definition:** Let \(M\) be a module. We say that \(M\) has the property \(\delta-(CE)\) if \(M\) has a \(\delta\)-supplement in every cofinite extension.

**Definition:** Let \(M\) be a module. We say that \(M\) has the property \(\delta-(CWE)\) if \(M\) has a weak \(\delta\)-supplement in every cofinite extension and \(M\) has the property \(\delta-(CWEE)\) if \(M\) has weak ample \(\delta\)-supplement in every cofinite extension.

**Proposition:** Every simple module has the property \(\delta-(CWE)\).

**Proof:** Let \(S\) be a simple module and \(N\) be any cofinite extension of \(S\). Then \(S\) is either a direct summand of \(N\) or \(\delta\)-small in \(M\). In the first case \(S \oplus S' = N\) for a submodule \(S' \leq N\) and so \(S'\) is a weak \(\delta\)-supplement of \(S\) in \(N\). In the second case, \(N\) is a weak \(\delta\)-supplement of \(S\) in \(N\). So in each case \(S\) has a weak \(\delta\)-supplement in \(N\). Finally \(S\) has the property \(\delta-(CWE)\).

It is easy to see that every module with the property \((CWE)\) and \(\delta-(CE)\) has the property \(\delta-(CWE)\). Let consider the \(\mathbb{Z}\)-module \(\mathbb{Z}\) and \(\mathbb{Z}\)-module \(Q\). Each of them is an example of a module that has the property \(\mathbb{Z}\)-module. It is natural to pose the question whether there exists similar result fort he properties of \(\delta-(CE)\) and \(\delta-(CE)\). To answer this, at the end of this section we shall give an example of a module which has the property \(\delta-(CWE)\) but not \(\delta-(CE)\).

Zöschinger proved in [15] that a module has the property \((EE)\) if and only if every submodule has the property \((E)\). Now we adopt only one side of this fact for our modules.
**Theorem:** Let $M$ be a module. If every submodule of $M$ has the property $\delta$-($\text{CWE}$), then $M$ has the property $\delta$-($\text{CWE}$).

**Proof:** Suppose that every submodule of $M$ has the property $\delta$-($\text{CWE}$). For a cofinite extension $N$ of $M$, let $N = M + K$ for some submodule $K$ of $N$. Then $N/M \cong K/(M \cap K)$ is finitely generated and so $M \cap K$ is a cofinite submodule of $K$. By the hypothesis, there exists a submodule $V$ of $K$ such that $K = (M \cap K) + V$ and $(M \cap K) \cap V = M \cap V \ll \delta K$. Note that $N = M + V$. It follows that $V$ is a weak $\delta$-supplement of $M$ in $N$. So $M$ has the property $\delta$-($\text{CWE}$).

In the following proposition we show that the property $\delta$-($\text{CWE}$) is preserved by direct summands.

**Proposition:** Every direct summand of a module with the property $\delta$-($\text{CWE}$) has the property $\delta$-($\text{CWE}$).

**Proof:** Let $N$ be a direct summand of $M$. Then there exists a submodule $K$ of $M$ such that $M = N \oplus K$. Let $L$ be a cofinite extension of $N$, $T$ be the external direct sum $L \oplus K$ and $\gamma: M \to T$ be the canonical embedding. Then $M \cong \gamma(M)$ has the property $\delta$-($\text{CWE}$). We have $L/N \cong (L \oplus K)/\gamma(M)$ is finitely generated. Since $\gamma(M)$ has the property $\delta$-($\text{CWE}$), then there exists a submodule $U$ of $T$ such that $T = \gamma(M) + U$ and $\gamma(M) \cap U \ll \delta T$. Consider the projection $\pi: T \to L$. By this way, we have $N + \pi(U) = L$. Also $(\pi) \leq \gamma(M)$, $\pi(\gamma(M) \cap U) \leq \pi(\gamma(M)) \cap \pi(U) = N \cap \pi(U) \ll \delta \pi(T) = L$. Therefore $\pi(U)$ is a weak $\delta$-supplement of $N$ in $L$.

Now by using the property $\delta$-($\text{CWE}$) we give a characterization for $\delta$-semilocal rings which is related to cofinitely weak $\delta$-supplemented modules investigated in [3, 8].

**Theorem:** Let $R$ be a ring. Then the following statements are equivalent:

a) $R$ is a $\delta$-semilocal ring.

b) Every $R$-module has the property $\delta$-($\text{CWE}$).

**Proof:** Let $R$ be a $\delta$-semilocal ring, $M$ be an $R$-module and $N$ be a cofinite extension of $M$. Since $R$ is $\delta$-semilocal, $N$ is a cofinitely weak $\delta$-supplemented module from [3]. Therefore $M$ has a weak $\delta$-supplement in $N$ as a submodule of $M$. Conversely, let $M$ be an $R$-module and $U$ be any cofinite submodule of $M$. By hypothesis, $U$ has the property $\delta$-($\text{CWE}$). Then $U$ has a weak $\delta$-supplement in $M$, so that $M$ is cofinitely weak $\delta$-supplemented. Hence $R$ is $\delta$-semilocal by [3].

**Corollary:** Let $R$ be a ring. Then every $R$-module is cofinitely weak $\delta$-supplemented if and only if every $R$-module has the property $\delta$-($\text{CWE}$).

Let $M$ be a module and $U$ be a submodule of $M$. If the factor module $M/U$ has the property $\delta$-($\text{CWE}$) $M$ does not need to have the property $\delta$-($\text{CWE}$). For example, for the ring $R = \mathbb{Z}$, the $R$-module $M = 2\mathbb{Z}/4\mathbb{Z}$ has a weak $\delta$-supplement in every cofinite extension since it is simple. But $2\mathbb{Z}$ does not have a weak $\delta$-supplement in its cofinite extension $\mathbb{Z}$.

Now we show that the statement mentioned above is true under a special condition.

**Proposition:** Let $M$ be a module and $U$ be a submodule of $M$. If $U \ll \delta M$ and the factor module $M/U$ has the property $\delta$-($\text{CWE}$), then $M$ has the property $\delta$-($\text{CWE}$).

**Proof:** Let $N$ be any extension of $M$. Since $M/U$ has the property $\delta$-($\text{CWE}$), there exists a submodule $V/U$ of $N/U$ such that $M/U + V/U = N/U$ and $(M \cap V)/U \ll \delta N/U$. Note that
Suppose that \((M \cap V) + S = N\) for a submodule \(S\) of \(N\) with \(N/S\) singular. Then we obtain \((M \cap V)/U + (S + U)/U = N/U\). Since \((M \cap V)/U \ll_{\delta} N/U\) and \(N/(S + U) \cong (N/S)/(S + U)/S\) is singular, we have that \((S + U)/U = N/U\). It follows that \(N = S + U = S\) and so \(M \cap V \ll_{\delta} N\) is obtained.

**Corollary:** Every \(\delta\)-local module has the property \(\delta\)-(CWE).

**Corollary:** Let \(M\) be a module. If \(M\) has the property \(\delta\)-(CWE), then so does every \(\delta\)-small cover of \(M\).

In [2], Çalışıcı and Türkmen defined cofinitely injective modules, that is, a module \(M\) is called cofinitely injective if every \(M\) is a direct summand of every cofinite extension.

Recall that a ring \(R\) is called left \(\delta\)-\(V\)-ring if \(\delta(M) = 0\) for every left \(R\)-module \(M\) [12].

**Proposition:** Let \(R\) be a left \(\delta\)-\(V\)-ring. An \(R\)-module \(M\) has the property \(\delta\)-(CWE) if and only if \(M\) is cofinitely injective.

**Proof:** Let \(M\) has the property \(\delta\)-(CWE) and \(N\) be any extension of \(M\). Then \(M\) has a weak \(\delta\)-supplement \(V\) in \(N\). We have \(M + V = N\) and \(M \cap V \ll_{\delta} N\). Hence \(M \cap V \leq \delta(N) = 0\) and so \(N = M \oplus V\). Conversely, let \(M\) be injective and \(N\) be any extension of \(M\). Then there exists a submodule \(K\) of \(N\) such that \(N = M \oplus K\). Hence \(K\) is a weak \(\delta\)-supplement of \(M\) in \(N\).

**Corollary:** Let \(R\) be a left \(\delta\)-\(V\)-ring. An \(R\)-module \(M\) has the property \(\delta\)-(CWE) if and only if \(M\) has the property \(\delta\)-(CE).

Since every submodule of a \(\delta\)-hollow module is \(\delta\)-small we can give the following proposition fort he completeness.

**Proposition:** If \(M\) is a \(\delta\)-hollow module, then \(M\) has the property \(\delta\)-(CWE).

**Proof:** Clear

Recall that over a left hereditary ring every factor module of an injective module is injective. In the following proposition, we show that every factor module of a module that has the property \(\delta\)-(CWE) over a left hereditary ring has the property \(\delta\)-(CWE).

**Proposition:** Let \(R\) be a left hereditary ring and \(M\) be a finitely generated module. If \(M\) has the property \(\delta\)-(CWE), then so does every factor module of \(M\).

**Proof:** For any submodule \(U\) of \(M\), let \(N\) be a cofinite extension of \(M/U\). Then \(N\) is finitely generated. By \(E(M)\), we denote the injective hull of \(M\). Since \(R\) is left hereditary, \(E(M)/U\) is injective, and so there exists a commutative diagram with exact rows in the following:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & U & \rightarrow & M & \rightarrow & M/U & \rightarrow & 0 \\
  &  \downarrow{id} &  \downarrow{f} & \downarrow{\varphi} & \downarrow{\sigma} & \downarrow{i_2} & & & \\
0 & \rightarrow & U & \rightarrow & K & \rightarrow & N & \rightarrow & 0 \\
\end{array}
\]
i.e., $\varphi = i_2 \pi$, where $\varphi : M \to K$ is a monomorphism. It follows that $K/\varphi(M) \cong K/\mathcal{C}(\alpha) \cong N$. Since $M$ has the property $\delta(CWE)$, $\varphi(M)$ has a weak $\delta$-supplement $V$ in $K$. So we obtain that $\sigma(V)$ is a weak $\delta$-supplement of $M/U$ in $N$. Hence $M/U$ has the property $\delta(CWE)$.

It is easy to see that every module that has the property $\delta(CWE)$ also has the property $\delta(CE)$. Now we give the following example to show that the converse statement may not be true in general.

**Example (see in [1]):** For primes $p$ and $q$, consider the ring $R := \mathbb{Z}_{pq} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, (p, b) = (q, b) = 1 \right\}$. $R$ is a $\delta$-semilocal ring that is not $\delta$-semiperfect. Then there exists an $R$-module $M$ that does not have the property $\delta(CE)$. But since $R$ is a $\delta$-semilocal ring, $M$ has the property $\delta(CWE)$.

In the following theorem we see a kind of a module that coincide the concepts of properties $\delta(CE)$ and $\delta(CWE)$ over it.

**Theorem:** Let $M$ be a semisimple module. Then the following statements are equivalent:

a) $M$ has the property $\delta(CE)$.

b) $M$ has a $\delta$-supplement in every cofinite extension $N$ that is a direct summand of $N$.

c) $M$ has the property $\delta(CWE)$.

**Proof:** ($a \implies b$): Let $N$ be any cofinite extension of $M$. By (a), we have $N = M + K$ and $M \cap K \ll_{\delta} K$ for some submodule $K \subseteq N$. Since $M$ is a semisimple module, then there exists a submodule $X$ of $M$ such that $M = (M \cap K) \oplus X$. So $(M \cap K) \cap X = K \cap X = 0$. Therefore $N = M + K = [(M \cap K) \oplus X] = K \oplus X$. This means $K$ is a $\delta$-supplement of $M$ that is a direct summand in $N$.

($b \implies c$): Clear.

($c \implies a$): Let $N$ be any cofinite extension of $M$. By (c), there exists a submodule $K$ of $N$ provided $N = M + K$ and $M \cap K \ll_{\delta} N$. Since $M \cap K \subseteq M$ and $M$ is semisimple there exists a submodule $T$ of $M$ such that $(M \cap K) \oplus T = M$.

So, $N = M + K = (M \cap K) \oplus T + K = K \oplus T$ is obtained. Since $K$ is a direct sum of $N$ and $M \cap K \ll_{\delta} N$, it is obtained that $M \cap K \ll_{\delta} K$.

**REFERENCES**


