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GENERALIZATIONS AND REFINEMENTS FOR BERGSTRÖM AND RADON'S INEQUALITIES

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Abstract. In the present work there are pointed and demonstrated some generalizations and refinements for Bergström and Radon's inequalities. But not before making some historical remaks on the parenthood of these inequalities. We present a new demonstration and a refinement for Radon's inequality, which is based on a recently initiated method, using the monotony of a sequence associated to the inequality. Some applications are also presented.
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It is well-known and very often used lately- *Bergström's inequality* (see [7] , [11] , [14]) , namely ,

1. Proposition (Bergström's inequality)
If $x_k \in R$, $a_k > 0, k \in \{1, 2, ..., n\}$, then the following inequality holds ,

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \ldots x_n)^2}{a_1 + a_2 + \ldots a_n}, \tag{1}$$

with equality for : $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.
It is equivalent with *Cauchy-Buniakowski-Schwarz inequality* **.**
For the less simple implication, *Bergström inequality* $\Rightarrow$ *C-B-S inequality*, see [2], [5], [20].
This inequality is often called *Titu Andreescu's inequality* (or *Andreescu* lemma –presented in [1] , having as base a problem published by the first author in the RMT journal, in 1979), or *Engel's inequality* (or *Cauchy-Schwarz inequality* in *Engel* form - in Germanofon mathematical literature , [12]).

In fact , this inequality , for the case $\boldsymbol{n = 2}$ was enounced by H. Bergström in 1949 , in the more general frame of complex number modules , from denominators and in more relaxed conditions , for nominators (see [7] , [14] , [11]) :

• Let $z_1, z_2 \in \mathbf{C}$ and $u, v \in \mathbf{R}$ such that $u \neq 0,\ v \neq 0,\ u + v \neq 0$.
Then we have:

$$i) \quad \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} \geq \frac{|z_1 + z_2|^2}{u + v} \quad , \quad if\ \frac{1}{u} + \frac{1}{v} > 0 \quad ; \tag{2}$$

$$ii) \quad \frac{|z_1|^2}{u} + \frac{|z_2|^2}{v} \leq \frac{|z_1 + z_2|^2}{u + v} \quad , \quad if\ \frac{1}{u} + \frac{1}{v} < 0 \quad . \tag{3}$$

The equality holds if and only if $\frac{z_1}{u} = \frac{z_2}{v}$.
More than that , the inequality (1) , is a particular case of some of *Radon's inequality*, discovered ever since 1913 , (see [19] , [9] and rediscovered (?..) in [16] and [6]) .

2. Proposition (Radon's inequality)
If $a_k, x_k > 0, p > 0, k \in \{1, 2, ..., n\}$, then the following inequality holds,

$$\sum_{k=1}^{n}\frac{x_k^{p+1}}{a_k^p}\geq\frac{\left(\sum\limits_{k=1}^{n}x_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{n}a_k\right)^{p}}, \tag{4}$$

with equality for : $\frac{x_1}{a_1}=\frac{x_2}{a_2}=\ldots=\frac{x_n}{a_n}$.

Clearly , $p=1$ - Bergström's inequality is obtained .

There are known some demonstrations of *Radon's inequality*, by using *Hölder's inequality* ([9], [16]) , or by using the mathematical induction, [6] . In what is to follow , we are going to demonstrate Radon's inequality through a method recently initiated in [13] , which uses the monotony of an associated sequence.

Proof

Let the sequence , $d_n:=\frac{x_1^{p+1}}{a_1^p}+\frac{x_2^{p+1}}{a_2^p}+\ldots+\frac{x_n^{p+1}}{a_n^p}-\frac{(x_1+x_2+\ldots+x_n)^{p+1}}{(a_1+a_2+\ldots+a_n)^p}$,

for which we are going to prove that $d_n\geq 0$, for any $n\geq 2$. For this we are going to demonstrate something more, namely that $(d_n)_n$ is an increasing monotonous sequence .

Indeed , we have ,

$$d_{n+1}-d_n=\sum_{k=1}^{n+1}\frac{x_k^{p+1}}{a_k^p}-\frac{\left(\sum\limits_{k=1}^{n+1}x_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{n+1}a_k\right)^{p}}-\sum_{k=1}^{n}\frac{x_k^{p+1}}{a_k^p}+\frac{\left(\sum\limits_{k=1}^{n}x_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{n}a_k\right)^{p}}=$$

$$=\frac{\left(\sum\limits_{k=1}^{n}x_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{n}a_k\right)^{p}}+\frac{x_{n+1}^{n+1}}{a_{n+1}^p}-\frac{\left(\sum\limits_{k=1}^{n+1}x_k\right)^{2}}{\sum\limits_{k=1}^{n+1}a_k}\geq 0 \quad .$$

For the last inequality , Radon's inequality has been used , for $n=2$,

$$\frac{\alpha^{p+1}}{a^p}+\frac{\beta^{p+1}}{b^p}\geq\frac{(\alpha+\beta)^{p+1}}{(a+b)^p}, \tag{5}$$

with : $\alpha=\sum\limits_{k=1}^{n}x_k$, $\beta=x_{n+1}$, $a=\sum\limits_{k=1}^{n}a_k$, $b=a_{n+1}$.

(For the demonstration of the inequality (5) , see [6]) .

It results that ,

$$d_n\geq d_{n-1}\geq ...\geq d_2\geq d_1=0. \tag{6}$$

3. Application If $a,b,c\in \mathrm{R}_+$, then ,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1 \quad . \tag{7}$$

(*The 42nd OIM, Washington D.C., 2001, Problem 2*)

We write the left member of the inequality under the form ,

$$M_s:=\frac{a^{\frac{3}{2}}}{\sqrt{a^3+8abc}}+\frac{b^{\frac{3}{2}}}{\sqrt{b^3+8abc}}+\frac{c^{\frac{3}{2}}}{\sqrt{c^3+8abc}}$$

and Radon's inequality is applied for $n = 3$,

$$\frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + \frac{x_3^{p+1}}{a_3^p} \geq \frac{(x_1 + x_2 + x_3)^{p+1}}{(a_1 + a_2 + a_3)^p},$$

with the substitutions: $x_1 \to a$, $x_2 \to b$, $x_3 \to c$; $a_1 \to a^3+8abc$, $a_2 \to b^3+8abc$, $a_3 \to c^3+8abc$ and p =1/2.

It is obtained,

$$M_s \geq \frac{(a+b+c)^{\frac{3}{2}}}{(a^3+b^3+c^3+24abc)^{\frac{1}{2}}} = \sqrt{\frac{(a+b+c)^3}{a^3+b^3+c^3+24abc}} \geq 1.$$

The last inequality is reduced – after some simple calculations , to the obvious inequality , $a(b^2+c^2) + a(b^2+c^2) + a(b^2+c^2) \geq 6abc$.

The demonstration method given previously and in [13], also underlines an interesting method of refining the inequalities, which we can also be seen in the following theorem,

4. Theorem (for refinement of Radon's inequality)

For $a_k, x_k > 0, p \geq 1, k \in \{1, 2, ..., n\}$, $n \in \mathrm{N}_{\geq 2}$, the inequality takes place,

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{\left(\sum\limits_{k=1}^{n} x_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{n} a_k\right)^{p}} + \max_{1 \leq i < j \leq n} \left(\frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i + x_j)^{p+1}}{(a_i + a_j)^p} \right), \tag{8}$$

with equality if and only if, $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.

Proof

As in the inequality sequence (6) , $d_1 = 0$, it only remains significant the inequality $d_n \geq d_2$, $(\forall) n \in \mathrm{N}_{\geq 2}$.

But,

$$d_2 = \frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} - \frac{(x_1 + x_2)^{p+1}}{(a_1 + a_2)^p},$$

therefore ,

$$d_n \geq \frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} - \frac{(x_1 + x_2)^{p+1}}{(a_1 + a_2)^p}, \quad (\forall)\ n \in N_{\geq 2}.$$

In the end, because of $\boldsymbol{d}_n$'s symmetry relatively to $\boldsymbol{a}_i$, and $\boldsymbol{x}_j$ variables, $i, j \in \{1, 2, ..., n\}$, it results that $d_n \geq \frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i+x_j)^{p+1}}{(a_i+a_j)^p}$, $(\forall) n \in \mathrm{N}_{\geq 2}$, $(\forall) i, j \in \{1, 2, ..., n\}$, hence the enounced relation holds. The equality condition $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$, is a necessary and sufficient condition for the equality in (4) as well as for the cancellation of the quantity

$$\max_{1 \leq i < j \leq n} \left(\frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i + x_j)^{p+1}}{(a_i + a_j)^p} \right).$$

For p =1 , a result proven in [13] is obtained.

5. Corollary (refinement of Bergström's inequality)

For $x_k \in R, a_k > 0, k \in \{1, 2, ..., n\}$, $n \in \mathrm{N}_{\geq 2}$, the inequality holds ,

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \ldots x_n)^2}{a_1 + a_2 + \ldots a_n} + \max_{1 \leq i < j \leq n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j \cdot (a_i + a_j)}, \tag{9}$$

with equality if and only if , $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.

6. Remark

The result from Theorem 4, Corollary 5 respectively, also forms a generalization of a contest problem from [18] .

Indeed , for $x_k = 1$ and $a_k \to x_k$, the enounce is obtained. Being $n \geq 2$ a natural number and x_1 , x_2 ,...,$x_n > 0$. Then :

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} - \frac{n^2}{x_1 + x_2 + \ldots x_n} \geq \max_{1 \leq i < j \leq n} \frac{(x_i - x_j)^2}{x_i x_j \cdot (x_i + x_j)} \quad , \tag{10}$$

For the demonstration of the next result we need the following,

7. Lemma

For $m \in \mathbf{R}_{\geq 1}$, $n \in \mathbf{N}^*$ and $x_i > 0$, then ,

$$\sum_{k=0}^{n} x_k^m \geq \frac{1}{n^{m-1}} \cdot \left(\sum_{k=1}^{n} x_k \right)^m \tag{11}$$

Proof

The inequality comes from the inequality between the power-means ([8], [9], [15]), namely, if $r, s \in$ R , $r \geq s$, then the inequality takes place,

$$\left(\frac{x_1^r + x_2^r + \ldots + x_n^r}{n} \right)^{1/r} \geq \left(\frac{x_1^s + x_2^s + \ldots + x_n^s}{n} \right)^{1/s} . \tag{12}$$

For $r = m$ and s =1 , the result is obtained .

8. Theorem (the generalization of Radon's inequality)

If $a_k, x_k > 0, p > 0, q \geq 1, k \in \{1, 2, ..., n\}$, then the inequality takes place,

$$\sum_{k=1}^{n} \frac{x_k^{p+q}}{a_k^p} \geq \frac{1}{n^{q-1}} \cdot \frac{\left(\sum\limits_{k=1}^{n} x_k \right)^{p+q}}{\left(\sum\limits_{k=1}^{n} a_k \right)^p}, \tag{13}$$

with equality for: $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.

Proof

Using Radon's inequality and the previous lemma, we successively have:

$$\sum_{k=1}^{n} \frac{x_k^{p+q}}{a_k^p} = \sum_{k=1}^{n} \frac{\left(x_k^{\frac{p+q}{p+1}} \right)^{p+1}}{a_k^p} \overset{(4)}{\geq} \frac{\left(\sum\limits_{k=1}^{n} x_k^{\frac{p+q}{p+1}} \right)^{p+1}}{\left(\sum\limits_{k=1}^{n} a_k \right)^p} \overset{(11)}{\geq}$$

$$\overset{(11)}{\geq} \frac{\left[\frac{1}{n^{\frac{p+q}{p+1} - 1}} \cdot \left(\sum\limits_{k=1}^{n} x_k \right)^{\frac{p+q}{p+1}} \right]^{p+1}}{\left(\sum\limits_{k=1}^{n} a_k \right)^p} = \frac{1}{n^{q-1}} \cdot \frac{\left(\sum\limits_{k=1}^{n} x_k \right)^{p+q}}{\left(\sum\limits_{k=1}^{n} a_k \right)^p}$$

For $q = 1$ in *Theorem 8*, *Radon's inequality* is obtained, and for $p = q = 1$, *Bergström's inequality* is obtained .

9. Corollary (the generalization of Radon's inequality - a variant) If $a_k, x_k > 0, , k \in \{1, 2, ..., n\}, p > 0, r \geq p + 1$, then the inequality holds,

$$\sum_{k=1}^{n} \frac{x_k^r}{a_k^p} \geq \frac{1}{n^{r-p-1}} \cdot \frac{\left(\sum_{k=1}^{n} x_k\right)^r}{\left(\sum_{k=1}^{n} a_k\right)^p}, \qquad (14)$$

with equality for : $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}$.

Proof

Noting $r := p + q$ in Theorem 8, $r \geq p+1$ results , and $q - 1 = r - p - 1$, hence the enounce . A similar result to the one in relation (14) is obtained in [17], using Jensen's inequality.

10. Application

If a, b, c are the sides of a triangle and $r \geq 2$, then ,

$$\frac{a^r}{b+c-a} + \frac{b^r}{c+a-b} + \frac{c^r}{a+b-c} \geq \frac{(a+b+c)^{r-1}}{3^{r-2}} \quad , \qquad (15)$$

or with the triangle known notations, we have ,

$$\frac{a^r}{p-a} + \frac{b^r}{p-b} + \frac{c^r}{p-c} \geq \frac{2^{r-1}}{3^{r-2}} \cdot p^{r-1} \quad . \qquad (16)$$

Using the above inequality extensions, numberless other inequalities , such as those in : [1] , [2] , [3] , [16] , [17] – can be proved or generalized .

New ones can also be obtained .

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