ORIGINAL PAPER

A NEW GEOMETRIC INEQUALITY AND ITS APPLICATIONS

JIAN LIU

East China Jiaotong University, 330013, Nanchang City, Jiangxi Province, China

Abstract. In this paper, we establish a ternary quadratic geometric inequality involving a triangle and a point by using the polar moment of the inertia inequality of M.S. Klamkin. We also give new result some applications, and propose several related conjectures which are checked by the computer.

Keywords: triangle, point, the polar moment of the inertia inequality, real number, inequality.

1. INTRODUCTION

In 1975, M. S. Klamkin [1] established the following geometric inequality: Let $ABC$ be a triangle with sides $a = BC$, $b = CA$, $c = AB$, and let $P$ be an arbitrary point, the distances of $P$ from the vertices $A$, $B$, $C$ are $R_1$, $R_2$, $R_3$ respectively. Then the inequality

$$ (x + y + z)(xR_1^2 + yR_2^2 + zR_3^2) \geq yza^2 + xzb^2 + xyC^2, \quad (1) $$

holds for all real numbers $x$, $y$, $z$. Equality if and only if $P$ lies the plane of $\triangle ABC$ and $x: y: z = \overline{S}_{ABC} : \overline{S}_{APC} : \overline{S}_{APB}$ ($\overline{S}_{ABC}$ denotes the algebra area etc.)

Inequality (1) is called the polar moment of the inertia inequality by M. S. Kalmkin. It is one of the most important results of geometric inequalities for the triangle, and there exist many consequences and applications for it, e.g., see [1-6]. The author has also researched this inequality. In 1992, we found it can be deduced the weighted inequality for the sides $a$, $b$, $c$ of the triangle (see [5]):

$$ \frac{s - a}{x} + \frac{s - b}{y} + \frac{s - c}{z} \geq \frac{(xa + yb + zc)s}{yza + xzb + xyC}, \quad (2) $$

where $s = (a + b + c)/2$ and $x$, $y$, $z$ are positive real numbers. Equality holds if and only if $x = y = z$. The inequality (2) was used to prove several geometric inequalities which connect with a point of a triangle in [5]. In 2007, the author [6] applied Klamkin inequality (1) and inversion transformation to establish the following geometric inequality with positive real numbers $x$, $y$, $z$:

$$ \frac{R_1^2}{x} + \frac{R_2^2}{y} + \frac{R_3^2}{z} \geq \frac{aR_1 + bR_2 + cR_3}{\sqrt{yza + xzb + xyC}}. \quad (3) $$

Equality holds if and only if $\triangle ABC$ is acute-angled, $P$ coincides with its orthocenter and $x: y: z = \cot A : \cot B : \cot C$. The author also gave inequality (3) some applications in [6].

The purpose of this paper is to establish the following ternary quadratic geometric inequality by using Klamkin’s polar moment of the inertia inequality.

Theorem: For any arbitrary point $P$ and real numbers $x$, $y$, $z$, we have

$$ \frac{R_1^2 + R_3^2}{a^2} + \frac{R_1^2 + R_2^2}{b^2} + \frac{R_1^2 + R_2^2}{c^2} \geq \frac{2}{3}(yz + zx + xy), \quad (4) $$

 Corresponding author: china99jian@163.com
with equality if and only if \( \triangle ABC \) is equilateral and \( P \) is its center.

In the next section we will prove the theorem. In the third section, we will give some applications of inequality (4). In the last section, several related conjectures are put forward.

2. THE PROOF OF THE THEOREM

**Proof.** To simplify matters. We denote cyclic sum by \( \Sigma \), then inequality (4) is

\[
\sum \left( \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) R_i^2 \geq \frac{2}{3} \sum yz.
\]  

(5)

According to Klamkin inequality (1), we have

\[
2 \sum \frac{x^2}{a^2} \sum \left( \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) R_i^2 \geq \sum \left( \frac{z^2}{c^2} + \frac{x^2}{a^2} \right) \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) a^2
\]  

(6)

If \( \sum yz < 0 \), then (5) holds clearly. If \( \sum yz > 0 \), from inequality (6), in order to prove (5) we need to prove that

\[
\sum \left( \frac{z^2}{c^2} + \frac{x^2}{a^2} \right) \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) a^2 \geq \frac{4}{3} \sum yz \sum \frac{x^2}{a^2}.
\]  

(7)

By replacing \( x \to xa, y \to yb, z \to zc \), the above inequality is equivalent to

\[
\sum (z^2 + x^2)(x^2 + y^2) a^2 \geq \frac{4}{3} \sum yzbc \sum x^2,
\]  

(8)

Namely

\[
3 \sum (z^2 + x^2)(x^2 + y^2) a^2 - 4 \sum yzbc \sum x^2 \geq 0
\]  

(9)

That is

\[
a_1a^2 + b_1a + c_1 \geq 0,
\]  

(10)

where

\[
a_1 = 3(z^2 + x^2)(x^2 + y^2),
\]

\[
b_1 = -4(yz + yb)x,
\]

\[
c_1 = 3 \left[ (x^2 + y^2) b^2 + (z^2 + x^2) c^2 \right] - 4 yzbc \left( x^2 + y^2 + z^2 \right).
\]

We rewrite \( c_1 \) as follow:

\[
c_1 = \left[ 3(b^2 + c^2)(y^2 + z^2) - 4 yzbc \right] x^2 + \left( y^2 + z^2 \right) \left( 3b^2 y^2 + 3c^2 z^2 - 4 yzbc \right).
\]

So, it is easy to see that \( c_1 > 0 \). Thus to prove (10) we need to show that \( 4a_1c_1 - b_1^2 \geq 0 \). However, we can obtain the following identity:
A new geometric inequality and its applications

Jian Liu

\[ 4a_1c_2 - b_2^2 = a_2b^2 + b_2b + c_1, \]  
\[ \text{where} \]
\[ a_2 = 20x^6y^2 + 36x^6z^2 + 40x^4y^4 + 76x^4y^2z^2 + 36x^4z^4 + 20x^2y^6 + 76x^2y^4z^2 + 56x^2y^2z^4 + 36y^6z^2 + 36y^4z^4 \]
\[ b_2 = -16c\left(5x^4 + 5x^2y^2 + 5x^2z^2 + 3y^2z^2\right)(x^2 + y^2 + z^2)y, \]
\[ c_2 = 4c^2(9x^6y^2 + 5x^6z^2 + 9x^4y^4 + 19x^4y^2z^2 + 10x^4z^4 + 14x^2y^4z^2 + 19x^2y^4z^4 + 5x^2z^6 + 9y^4z^4 + 9y^2z^6). \]

Since again \( a_2 > 0, c_2 > 0 \), it suffices to prove that \( 4a_2c_2 - b_2^2 \geq 0 \). But

\[ 4a_2c_2 - b_2^2 = 
\]
\[ = 192c^2\left(z^2 + x^2\right)(x^2 + y^2). \]
\[ \left[ 30\sum y^6z^6 + 15\sum y^4z^4(y^4 + z^4) - 18x^2y^2z^2\sum(y^2 + z^2)x^4 + 2x^2y^2z^2\sum x^6 - 78x^4y^4z^4 \right] \]

Hence, we only need to prove the following inequality:

\[ 30\sum v^3w^3 + 15\sum v^2w^2(v^2 + w^2) - 18uvw\sum(v + w)u^2 + 2uvw\sum u^3 - 78u^2v^2w^2 \geq 0 \]  
holds for non-negative real numbers \( u, v, w \). Denote the left hand side of (13) by \( Q \), after analyzing we obtain the following identity:

\[ Q = Q_1 + Q_2 + Q_3, \]  
\[ \text{where} \]
\[ Q_1 = 15\sum v^3w^3(v - w)^2 + 12uvw\sum u(v - w)^2 \]
\[ Q_2 = 2uvw\left(\sum u^3 - 3uvw\right) + 30\left(\sum v^3w^3 - 3u^2v^2w^2\right) \]
\[ Q_3 = 30\sum v^2w^2(u - v)(u - w) \]

Obviously, inequalities \( Q_1 \geq 0, Q_2 \geq 0 \) hold true. In addition, by replacing \( u \to uv, v \to wu, w \to uv \)
in the simple case of the famous Schur inequality

\[ \sum u(u - v)(u - w) \geq 0, \]

we know \( \sum v^2w^2(u - v)(u - w) \geq 0 \), hence \( Q_3 \geq 0 \). Therefore, the inequality \( Q \geq 0 \) follows from (14).

It is easy to see that equality in (7) holds if and only if \( x = y = z \) and triangle \( ABC \) is equilateral. Further, the equality in (4) holds just as the mentions of our theorem. This completes the proof of the Theorem.

\[ \square \]
3. APPLICATIONS OF THE THEOREM

In this section, we will discuss some applications of our theorem.

In inequality (4), for \(x = \frac{a}{\sqrt{R_2^2 + R_3^2}}, \ y = \frac{b}{\sqrt{R_1^2 + R_3^2}}, \ z = \frac{c}{\sqrt{R_1^2 + R_2^2}}\), we obtain

\[bc \leq \sqrt{\left(\frac{R_2^2 + R_3^2}{R_1^2 + R_3^2}\right)\left(\frac{R_2^2 + R_3^2}{R_1^2 + R_3^2}\right)\left(\frac{R_2^2 + R_3^2}{R_1^2 + R_3^2}\right)} \leq \frac{9}{2}\]

Corollary 1: For \(\Delta ABC\) and arbitrary point \(P\), we have

(16)

From inequality (16), we can easily get two following corollaries again:

Corollary 2: For \(\Delta ABC\) and arbitrary point \(P\), we have

(17)

Corollary 3: For \(\Delta ABC\) and arbitrary point \(P\), we have

(18)

In 1996, the author [7] established the following decline theorem about the ternary quadratic inequality: Let \(p_1, p_2, p_3, q_1, q_2, q_3\) and \(m\) be positive real numbers. If the ternary quadratic inequality:

(19)

holds for arbitrary real numbers \(x, y, z\), then

(20)

where \(0 < k < m\). By the decline theorem and inequality (4), we get the more general result than (18):

Corollary 4: For \(\Delta ABC\) and arbitrary point \(P\), we have

(21)

Putting in (21) \(x = \sqrt{a}, \ y = \sqrt{b}, \ z = \sqrt{c}\), one has
Corollary 5: For $\triangle ABC$ and arbitrary point $P$, we have

$$\sqrt{R_a^2 + R_b^2} + \sqrt{R_b^2 + R_c^2} + \sqrt{R_c^2 + R_a^2} \geq \frac{2}{3} \left( \sqrt{bc} + \sqrt{ca} + \sqrt{ab} \right)$$

(22)

Let $\Omega_i$ denote one of the Crelle – Brocard point of $\Delta ABC$, then we have the following formula (see [4, P1279]):

$$A\Omega_i^2 = \frac{c^4 b^2}{b^2 c^2 + c^2 a^2 + a^2 b^2}$$

(23)

So, if we order $P = \Omega_i$ in (4), then it follows at once

$$\left( c^2 a^2 + b^4 \right) x^2 + \left( a^2 b^2 + c^4 \right) y^2 + \left( a^2 c^2 + c^4 \right) z^2 \geq \frac{2}{3} \left( yz + zx + xy \right) \left( b^2 c^2 + c^2 a^2 + a^2 b^2 \right)$$

(24)

By replacing $x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$, we get the weighted inequality for the sides of $\Delta ABC$:

Corollary 6: For $\Delta ABC$ and arbitrary numbers $x, y, z$, we have

$$\left( b^2 c^2 + a^4 \right) x^2 + \left( c^2 a^2 + b^4 \right) y^2 + \left( a^2 b^2 + c^4 \right) z^2 \geq \frac{2}{3} \left( yz + zx + xy \right) \left( b^2 c^2 + c^2 a^2 + a^2 b^2 \right)$$

(25)

We denote the sides $B'C', C'A', A'B'$ of $\Delta A'B'C'$ by $a', b', c'$, the distances form arbitrary point $P'$ to the vertices $A', B', C'$ by $D_1', D_2', D_3'$ respectively. In (4) we take

$$x = \frac{D_1'}{a'}, y = \frac{D_2'}{b'}, z = \frac{D_3'}{c'}$$

then using Hayashi inequality [8] (it can be deduced by Klamkin inequality (1), see [4]):

$$\frac{D_2 D_3}{b' c'} + \frac{D_1 D_3}{c' a'} + \frac{D_1 D_2}{a' b'} \geq 1$$

(26)

we obtain

Corollary 7: For $\Delta ABC$, $\Delta A'B'C'$ and two points $P$, $P'$, we have

$$\frac{\left( R_a^2 + R_b^2 \right) D_1^2}{(a')^2} + \frac{\left( R_b^2 + R_c^2 \right) D_2^2}{(b')^2} + \frac{\left( R_c^2 + R_a^2 \right) D_3^2}{(c')^2} \geq \frac{2}{3}$$

(27)

If we order $\Delta A'B'C' \equiv \Delta ABC$ and $P = P'$, then we get

Corollary 8: For $\Delta ABC$ and arbitrary point $P$, we have
Let $P$ lie in the plane of $\Delta ABC$, and let $D, E, F$ be the orthogonal projections of $P$ on the lines $BC, CA, AB$ respectively (See Fig. 1). Put $PD = r_1, PE = r_2, PF = r_3$, note that $EF = R_1\sin A, FD = R_2\sin B, DE = R_3\sin C$, inequality (4) is applied to pedal triangle $DEF$ to give

**Corollary 9**: If $P$ lies in the plane of $\triangle ABC$ and does not coincide with the vertices $A, B, C$, then the inequality

$$\frac{r_1^2 + r_2^2}{R_1^2 \sin^2 A} x^2 + \frac{r_1^2 + r_3^2}{R_2^2 \sin^2 B} y^2 + \frac{r_1^2 + r_3^2}{R_3^2 \sin^2 C} z^2 \geq \frac{2}{3}(yz + zx + xy)$$

holds for arbitrary real numbers $x, y, z$.

Obviously, inequality (29) is equivalent to the following:

$$\frac{r_1^2 + r_2^2}{R_1^2} x^2 + \frac{r_1^2 + r_2^2}{R_2^2} y^2 + \frac{r_1^2 + r_2^2}{R_3^2} z^2 \geq \frac{2}{3}(yz \sin B \sin C + zx \sin C \sin A + xy \sin A \sin B)$$

**Remark**: For $x = y = z = 1$ in (29) we get

$$\frac{r_1^2 + r_2^2}{R_1^2 \sin^2 A} + \frac{r_1^2 + r_2^2}{R_2^2 \sin^2 B} + \frac{r_1^2 + r_2^2}{R_3^2 \sin^2 C} \geq 2$$

When triangle $ABC$ is acute-angled, the author [9] has generalized the above inequality to the case involving two triangles:

$$\frac{r_1^2 + r_2^2}{R_1^2 \sin^2 A} + \frac{r_1^2 + r_2^2}{R_2^2 \sin^2 B} + \frac{r_1^2 + r_2^2}{R_3^2 \sin^2 C} \geq 2$$

where $A', B', C'$ are the angles of arbitrary triangle $A'B'C'$. Equality holds if and only if $\sin A : \sin^2 A' = \sin B : \sin^2 B' = \sin C : \sin^2 C'$ and $P$ is the incenter of acute triangle $ABC$.

In (29), we take $x = a, y = b, z = c$. Then using $bc \sin B \sin C = h_a^2$ ($h_a$ denotes the altitude of $BC$ etc.), we get
**Corollary 10:** If \( P \) lies in the plane of \( \triangle ABC \) and does not coincide with the vertices \( A, B, C \), then

\[
\frac{r_1^2 + r_2^2}{R_1^2} a^2 + \frac{r_1^2 + r_3^2}{R_2^2} b^2 + \frac{r_2^2 + r_3^2}{R_3^2} c^2 \geq \frac{2}{3}(h_a^2 + h_b^2 + h_c^2) \tag{33}
\]

In addition, for \( x = R_1, y = R_2, z = R_3 \) in (29), it follows that

**Corollary 11:** For any \( \triangle ABC \) and arbitrary point \( P \), we have

\[
\frac{r_1^2 + r_2^2 + r_3^2}{\sin^2 A} \geq \frac{2}{3}(R_2 R_3 + R_3 R_1 + R_1 R_2) \tag{34}
\]

In (29) we take \( x = \tan \frac{A}{2}, y = \tan \frac{B}{2}, z = \tan \frac{C}{2} \), then using well-known triangle identity:

\[
\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1, \tag{35}
\]

We get again the following:

**Corollary 12:** Let \( P \) be an arbitrary point which does not coincide with the vertices of \( \triangle ABC \), then

\[
\frac{r_1^2 + r_2^2}{R_1^2 \cos^4 \frac{A}{2}} + \frac{r_2^2 + r_3^2}{R_2^2 \cos^4 \frac{B}{2}} + \frac{r_3^2 + r_1^2}{R_3^2 \cos^4 \frac{C}{2}} \geq \frac{8}{3} \tag{36}
\]

In inequality (30) we put \( x = \frac{R_1}{a^2}, y = \frac{R_2}{b^2}, z = \frac{R_3}{c^2} \), it follows that

\[
\frac{r_1^2 + r_2^2}{a^4} + \frac{r_2^2 + r_3^2}{b^4} + \frac{r_3^2 + r_1^2}{c^4} \geq \frac{2}{3} \left( \frac{R_2 R_3}{b^2 c^2} \sin B \sin C + \frac{R_1 R_3}{c^2 a^2} \sin C \sin A + \frac{R_1 R_2}{a^2 b^2} \sin A \sin B \right) \tag{37}
\]

\[
= \frac{1}{6R^2} \left( R_1 R_2 \frac{1}{bc} + R_1 R_3 \frac{1}{ca} + R_2 R_3 \frac{1}{ab} \right) \geq \frac{1}{6R^2}
\]

(where \( R \) is the radius of \( \triangle ABC \)) the last step was obtained by using Hayashi inequality (26) for the \( \triangle ABC \) and point \( P \). Therefore, we have

**Corollary 13:** For any \( \triangle ABC \) and arbitrary point \( P \) in the plane of \( \triangle ABC \), the following inequality holds:

\[
\frac{r_1^2 + r_2^2}{a^4} + \frac{r_2^2 + r_3^2}{b^4} + \frac{r_3^2 + r_1^2}{c^4} \geq \frac{1}{6R^2} \tag{37}
\]
4. SEVERAL CONJECTURES

In the last section, we propose several related conjectures. The author has considered the stronger inequalities of Corollary 5, 8 and 11. After being checked by the computer, the following three conjectures are put forward respectively:

**Conjecture 1**: For $\triangle ABC$ and arbitrary point $P$ holds:

$$\sqrt{R_2^2 + R_3^2} + \sqrt{R_3^2 + R_1^2} + \sqrt{R_1^2 + R_2^2} \geq \frac{2}{3}(a + b + c)$$  \hfill (38)

**Conjecture 2**: For $\triangle ABC$ and arbitrary point $P$ holds:

$$\frac{(R_2 + R_3)R_1^2}{a^4} + \frac{(R_3 + R_1)R_2^2}{b^4} + \frac{(R_1 + R_2)R_3^2}{c^4} \geq \frac{4}{3}$$  \hfill (39)

**Conjecture 3**: For $\triangle ABC$ and arbitrary point $P$ holds:

$$\frac{r_2^2 + r_3^2}{\sin^2 A} + \frac{r_3^2 + r_1^2}{\sin^2 B} + \frac{r_1^2 + r_2^2}{\sin^2 C} \geq \frac{2}{9}(R_1 + R_2 + R_3)^2$$  \hfill (40)

Considering the exponent generalization of Corollary 12, we pose the conjecture:

**Conjecture 4**: Let $P$ be an arbitrary point which does not coincide with the vertices of $\triangle ABC$. If $k \geq 2$, then

$$\frac{r_2^2 + r_3^2}{R_i^2 \cos^k A} + \frac{r_3^2 + r_1^2}{R_i^2 \cos^k B} + \frac{r_1^2 + r_2^2}{R_i^2 \cos^k C} \geq \frac{2^{k-1}}{3^{k-1}}$$  \hfill (41)

REFERENCES


Manuscript received: 09.12.2010
Accepted paper: 06.01.2011
Published online: 01.02.2011