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SOME GEOMETRIC INEQUALITIES OF RADON-ERDŐS-MORDELL TYPE

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Abstract. Some Erdös-Mordell type inequalities for general convex polygons are presented. The main tool in the proofs is the Radon inequality.

Keywords: Erdös-Mordell type inequality, Radon inequality, convex polygon.

1. INTRODUCTION

The purpose of this article is to establish some geometric inequalities (other than [2]) on Erdös-Mordell type, in convex polygons, used J. Radon’s inequality (see for example[1]).

Let \( a, b, c, d, x_k, y_k \in R^*_+, \forall k = 1, n \) and \( X_n = \sum_{k=1}^{n} x_k, Y_n = \sum_{k=1}^{n} y_k \).

Theorem 1. (A generalization of J. Radon’s inequality).

If \( m, p, q, s \in R, r \in [1, \infty) \) such that \( cY_n^s > d \max_{1 \leq k \leq n} y_k^s, \forall k = 1, n \), then:

\[
\sum_{k=1}^{n} \left( aX_n^p + bx_k^q \right)^{m+1} x_k^{(m+1)} \geq \left( an^q X_n^{p+r} + bX_n^q Y_n^{m+1} \right)^{m+1} \cdot \frac{1}{n^{(m+1)(q+r-1)-m}}
\]

Proof: We denoted

\[
u_k = (aX_n^p + bx_k^q)x_k^s, \quad v_k = (cY_n^s - dy_k^s)y_k^s, \quad \forall k = 1, n, \quad V_n = \sum_{k=1}^{n} v_k
\]

and the LHS of (1) becomes:

\[
\sum_{k=1}^{n} \frac{u_k^{m+1}}{v_k^{m+1}} = \sum_{k=1}^{n} v_k \left( \frac{u_k}{v_k} \right)^{m+1} = V_n \sum_{k=1}^{n} v_k \left( \frac{u_k}{v_k} \right)^{m+1}
\]

Since the function \( f : R^*_+ \rightarrow R^*_+, f(x) = x^{m+1} \) is convex on \( R^*_+ \), we use Jensen’s inequality and we obtain that:

\[
\sum_{k=1}^{n} \frac{v_k}{V_n} \cdot f \left( \frac{u_k}{v_k} \right) \geq \sum_{k=1}^{n} f \left( \frac{u_k}{v_k} \right) = \sum_{k=1}^{n} \frac{u_k}{V_n} \cdot \frac{V_n}{V_n^{m+1}} = \sum_{k=1}^{n} \frac{v_k}{V_n} \cdot \frac{u_k}{v_k} \cdot \frac{V_n^{m+1}}{V_n^{m+1}}
\]

Therefore,

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\[
\sum_{k=1}^{n} \frac{u_k^{m+1}}{v_k^m} \geq V_n \cdot \left( \sum_{k=1}^{n} \frac{u_k}{v_k} \right)^{m+1} = \left( \sum_{k=1}^{n} \frac{u_k}{v_k} \right)^{m+1} \Rightarrow
\]
\[
\Rightarrow \sum_{k=1}^{n} \left( aX_n^p + bx_k^q \right)^{m+1} x_k^{(m+1)} \geq \left( \sum_{k=1}^{n} \frac{aX_n^p + bx_k^q}{cY_n - dy_k} \right)^{m+1} y_k^m \geq \left( \sum_{k=1}^{n} \frac{aX_n^p + bx_k^q}{cY_n - dy_k} \right)^{m+1} =
\]
\[
\left( \frac{aX_n^p + bx_k^q}{cY_n - dy_k} \right)^{m+1} y_k^m = \frac{\left( aX_n^p + bx_k^q \right)^{m+1}}{\left( cY_n^m - dy_k \right)^{m+1}} y_k^m
\]
Because the functions \( g, h, k : R^+_n \rightarrow R^+ \), \( g(x) = x^r, h(x) = x^{q+r}, k(y) = y^{s+1} \) are convex on \( R^+_n \), also by Jensen’s inequality we have:
\[
\sum_{k=1}^{n} x_k^r = \sum_{k=1}^{n} g(x_k) \geq \frac{1}{n} \cdot \sum_{k=1}^{n} x_k = n \cdot \frac{X_n^r}{n^r} = \frac{X_n^r}{n^r},
\]
\[
\sum_{k=1}^{n} x_k^{q+r} = \sum_{k=1}^{n} h(x_k) \geq \frac{1}{n} \cdot \sum_{k=1}^{n} x_k = n \cdot \frac{X_n^{q+r}}{n^{q+r}} = \frac{X_n^{q+r}}{n^{q+r}},
\]
\[
\sum_{i=1}^{n} y_i^{s+1} = \sum_{i=1}^{n} k(y_i) \geq \frac{1}{n} \cdot \sum_{i=1}^{n} y_i = n \cdot \frac{Y_n^{s+1}}{n^{s+1}} = \frac{Y_n^{s+1}}{n^{s+1}}.
\]
Then, we deduce that:
\[
\sum_{k=1}^{n} \left( aX_n^p + bx_k^q \right)^{m+1} x_k^{(m+1)} \geq \frac{\left( a \cdot \frac{X_n^p}{n^p} + b \cdot \frac{X_n^{q+r}}{n^{q+r}} \right)^{m+1}}{\left( cY_n^m - dy_k \right)^{m+1}} y_k^m = \frac{\left( aX_n^p + bx_k^q \right)^{m+1}}{\left( cY_n^m - dy_k \right)^{m+1}} y_k^m.
\]
and we are done. 

**Observation 1.1.** If \( p = q = s = 0 \), then (I) becomes:
\[
\sum_{k=1}^{n} \left( a + b \right)^{m+1} x_k^{(m+1)} \geq \left( a + b \right)^{m+1} \frac{1}{\left( c - d \right)^m y_k^m} \Rightarrow \sum_{k=1}^{n} x_k^{(m+1)} \leq \frac{\left( c - d \right)^m y_k^m}{\sum_{k=1}^{n} x_k^{(m+1)}} \geq \frac{X_n^r}{Y_n^m}
\]
If we consider \( r = 1 \), then by (I’) we obtain:
\[
\sum_{k=1}^{n} x_k^{m+1} \leq \frac{X_n^{m+1}}{Y_n^m}
\]
which is just the inequality of J. Radon, with equality if and only if there exists \( t \in R^+_n \) such that \( x_k = ty_k, \forall k = 1, n \).
Observation 1.2. If \( m = 1 \), then (1) becomes:

\[
\sum_{k=1}^{n} \left( aX_n^p + bx_n^q \right)^{r(m+1)} x_k^{2r} \geq \frac{\left( an^q X_n^{p+r} + bx_n^{q+r} \right)^{r(m+1)}}{\left( cn^s - d \right)^m X_n^{m(m+1)}}, \quad \frac{1}{n^{(m+1)q(r+1)-ms}} \quad (1^*)
\]

If we take \( p = q = s = 0, r = 1 \), then by (1”’) we obtain:

\[
\sum_{k=1}^{n} \frac{x_k^2}{y_k^{2r}} \geq \frac{X_n^2}{Y_n^{2r}} \quad \text{(B)}
\]

but that is just the inequality of H. Bergström.

Next, we consider \( A_1A_2\ldots A_n \) (\( n \geq 3 \)), a convex polygon and for any point \( M \) from inside the polygon we use the notations: \( x_k = MA_k \), \( y_k \) the distance from \( M \) to the line \( A_kA_{k+1} \), \( a_k \) the length of the side \([A_kA_{k+1}]\) of the polygon (\( \forall k = 1, n \)), \( 2p \) is the perimeter of the polygon, \( S \) is the area of the given polygon and \( A_{n+1} = A_1 \).

Theorem 2. If \( A_1A_2\ldots A_n \) (\( n \geq 3 \)) is a convex polygon where we use the above notations and

\[
a, b, c, d \in R^*, m, p, q, s \in R^*, r \in [1, \infty),
\]

such that

\[
cY_n^{r'} > d \max_{1 \leq k \leq n} y_k^{s'}, \quad \forall k = 1, n,
\]

then:

\[
\sum_{k=1}^{n} \left( aX_n^p + bx_n^q \right)^{r(m+1)} x_k^{r(m+1)} \geq \frac{\left( an^q X_n^{p+r} + bx_n^{q+r} \right)^{r(m+1)}}{\left( cn^s - d \right)^m X_n^{m(m+1)}}, \quad \frac{1}{n^{(m+1)q(r+1)-ms}} \quad (2)
\]

Proof: By L. Fejes Tóth’s inequality (see e.g. [2])

\[
\sum_{k=1}^{n} y_k = Y_n \leq \left( \cos \frac{\pi}{n} \right) \sum_{k=1}^{n} x_k = \left( \cos \frac{\pi}{n} \right) X_n,
\]

and by (1) we deduce what we have to show. \( \square \)

Observation 2.1. If we put \( p = q \), then by (2) we obtain:

\[
\sum_{k=1}^{n} \left( aX_n^p + bx_n^q \right)^{r(m+1)} x_k^{r(m+1)} \geq \frac{\left( an^q X_n^{p+r} + bx_n^{q+r} \right)^{r(m+1)}}{\left( cn^s - d \right)^m X_n^{m(m+1)}}, \quad \frac{1}{n^{(m+1)q(r+1)-ms}} \quad (2')
\]

If we consider \( p = s = 0 \), then by (2’) we deduce that:

\[
\sum_{k=1}^{n} \frac{x_k^{r(m+1)}}{y_k^{m}} \geq X_n^{m(r+1)+r} \quad \frac{\left( \cos \frac{\pi}{n} \right)^m}{n^{(m+1)q(r+1)-ms}} \quad (2'')
\]

If we take \( r = 1 \) then by (2’’) yields:

\[
\sum_{k=1}^{n} \frac{x_k^{m+1}}{y_k^{m}} \geq X_n^{m} \quad \frac{\left( \sec \frac{\pi}{n} \right)^m}{n^{(m+1)q(r+1)-ms}} \quad (2'')
\]

Remark 2.1. Putting \( m = 1 \) in (2’’) we obtain the relation (18) from [2].
Theorem 3. If we have the notations presented above, then
\[
\sum_{k=1}^{n} \frac{x_k^{m+1}}{p^{\frac{m+1}{m}}(y_k y_{k+1} \cdots y_{k+p-1})^m} \geq \frac{X_n^{m+1}}{p^{m-1}Y_n^m}, \forall m \in \mathbb{R}^+, p \in \mathbb{N}^* - \{1\}
\]
where \(y_{n+j} = y_j, \forall j = 0, p - 1\).

**Proof:** By AM-GM inequality we have:
\[
\left(\prod_{j=0}^{p-1} y_{k+j}\right)^{\frac{1}{m}} \leq \frac{1}{p} \sum_{j=0}^{p-1} y_{k+j}, \forall k = 1, n,
\]
which yields
\[
\sum_{k=1}^{n} \frac{x_k^{m+1}}{p^{\frac{m+1}{m}}(\prod_{j=1}^{p-1} y_{k+j})^m} \geq p \sum_{k=1}^{n} \frac{x_k^{m+1}}{\left(\sum_{j=0}^{p-1} y_{k+j}\right)^m},
\]
then by (R)
\[
\sum_{k=1}^{n} \frac{x_k^{m+1}}{\left(\prod_{j=1}^{p-1} y_{k+j}\right)^m} \geq p \cdot \frac{X_n^{m+1}}{p^{m-1}Y_n^m} = \frac{X_n^{m+1}}{p^{m-1}Y_n^m},
\]
which completes the proof. ■

**Observation 3.1.** By (3) and L. Fejes Tóth’s inequality we deduce that:
\[
\sum_{k=1}^{n} \frac{x_k^{m+1}}{\left(p^{\frac{m+1}{m}}(\prod_{j=1}^{p-1} y_{k+j})^m\right)} \geq \frac{X_n^{m+1}}{p^{m-1}X_n^{\cos\left(\frac{\pi}{2}\right)^m}} = \frac{X_n^{m+1}}{p^{m-1}\cos^{\frac{\pi}{2}}} = \frac{X_n^{m+1}}{p^{m-1}\cos^{\frac{\pi}{2}}}
\]
which is a Radon-Erdös-Mordel type inequality.

If we consider \(m = 1\), then by (3') we obtain that:
\[
\sum_{k=1}^{n} \frac{x_k^2}{\left(p^{\frac{1}{m}}(\prod_{j=0}^{p-1} y_{k+j})^m\right)} \geq X_n \sec \frac{\pi}{n}
\]
(3'')

For the same convex polygon \(A_1A_2\ldots A_n\) \((n \geq 3)\), \(A_{n+1} = A_1\) and for any point \(M\) of space which is not on the line \(A_kA_{k+1}\), we denoted by \(y_k(M)\) the distance from \(M\) to line \(A_kA_{k+1}\), \(s_k(M) = \text{area}[A_kMA_{k+1}], \ \forall k = 1, n\) and \(S(M) = \sum_{k=1}^{n} s_k(M)\).

**Theorem 4.** If \(M\) and \(N\) are the points in space which is not on the line \(A_kA_{k+1}\), \((\forall k = 1, n)\), then:
\[
\sum_{k=1}^{n} a_k \left(\frac{1}{y_k(M)^m} + \frac{1}{y_k(N)^m}\right) \geq 2p^{m-1} \left(\frac{(S(M))^m + (S(N))^m}{(S(M)S(N))^m}\right)
\]
(4)
Proof: We have:

\[
U_n = \sum_{k=1}^{n} a_k \left( \frac{1}{y_k^m(M)} + \frac{1}{y_k^m(N)} \right) = \sum_{k=1}^{n} a_k \left( \frac{1}{(a_k y_k^M)^m} + \frac{1}{(a_k y_k^N)^m} \right) = \sum_{k=1}^{n} a_k \left( \frac{1}{2^m S_k^m(M)} + \frac{1}{2^m S_k^m(N)} \right),
\]

where we apply (R) and yields

\[
U_n \geq \frac{1}{2^m} \left( \sum_{k=1}^{n} a_k \right)^{m+1} + \frac{1}{2^m} \left( \sum_{k=1}^{n} a_k \right)^{m+1} = \frac{(2p)^{m+1}}{2^m} \left( \frac{1}{(S(M))^m} + \frac{1}{(S(N))^m} \right) \geq 2p^{m+1} \left( S^m + S_1^m \right) \quad (4')
\]

where \( S \) is the area of the polygon with the perimeter \( 2p \), and \( S_1 \) is the lateral area of the pyramid with vertex \( N \) and base the polygon \( A_1A_2\ldots A_n \).

If \( M \) and \( N \) are inside of the polygon then \( S(M) = S(N) = S \), and (4') becomes:

\[
\sum_{k=1}^{n} a_k \left( \frac{1}{y_k^m(M)} + \frac{1}{y_k^m(N)} \right) \geq 4p^{m+1} \frac{S^m}{S^m} = 4p^{m+1} \quad (4'')
\]

If in addition \( M \equiv N \), then we obtain that:

\[
2 \sum_{k=1}^{n} a_k \left( \frac{1}{y_k^m(M)} \right) \geq 4p^{m+1} \quad \iff \quad \sum_{k=1}^{n} a_k \left( \frac{1}{y_k^m(M)} \right) \geq 2p^{m+1} \quad (4''')
\]

Remark 4.1. Taking \( m = 1 \), then by (4''') we deduce that:

\[
\sum_{k=1}^{n} a_k \frac{1}{y_k^m(M)} > \frac{2p^2}{S},
\]

i.e. the problem 10876 , proposed by D. Buşneag in G,M,-B nr. 1/1971, pp.35.

For the convex polygon \( A_1A_2\ldots A_n \) \( (n \geq 3) \), \( A_{n+1} = A_1 \) and \( M \) a point in space which is not on the line \( A_k A_{k+1} \), \( (\forall k \in \{1, n\}) \), we denote by \( m_k = \mu(\angle A_k MA_{k+1}) \), \( k \in \{1, n\} \), the measure in radians of the angle \( \angle A_k MA_{k+1} \), \( k \in \{1, n\} \).

Theorem 5. If \( A_1A_2\ldots A_n \) \( (n \geq 3) \) is a convex polygon as above, and \( M \) is a point in space which is not on the line \( A_k A_{k+1} \), \( (\forall k \in \{1, n\}) \), with \( m_k \in \left(0, \frac{\pi}{2}\right], k \in \{1, n\} \), then:
\[
\sum_{k=1}^{n} \frac{a_k^2}{x_k x_{k+1}} \geq 4n \sin^2 \frac{m_1 + m_2 + \ldots + m_n}{2n} \tag{5}
\]

**Proof:** In the triangle \(A_k MA_{k+1}\), by the Law of Cosines we have,
\[
a_k^2 = x_k^2 + x_{k+1}^2 - 2x_k x_{k+1} \cos(\angle A_k MA_{k+1}) = x_k^2 + x_{k+1}^2 - 2x_k x_{k+1} \cos m_k, \forall k = 1, n,
\]
where we apply AM-GM inequality and we obtain that:
\[
a_k^2 \geq 2x_k x_{k+1} (1 - \cos m_k) = 4x_k x_{k+1} \sin^2 \frac{m_k}{2}, \forall k = 1, n \iff \frac{a_k^2}{x_k x_{k+1}} \geq 4 \sin^2 \frac{m_k}{2}, \forall k = 1, n.
\]

Hence:
\[
\sum_{k=1}^{n} \frac{a_k^2}{x_k x_{k+1}} \geq \sum_{k=1}^{n} \frac{4 \sin^2 \frac{m_k}{2}}{2}.
\]

Since the function \(f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \sin^2 \frac{x}{2}\) is convex on \(\left[0, \frac{\pi}{2}\right]\), by Jensen’s inequality we have that:
\[
\sum_{k=1}^{n} \sin^2 \frac{m_k}{2} \geq n \sin^2 \frac{m_1 + m_2 + \ldots + m_n}{2n}.
\]

From the last two relations we get what must be demonstrated.

**Remark 5.1.** If \(n \geq 5\), \(M \in \text{Int} A_1 A_2 \ldots A_n\), then \(\sum_{k=1}^{n} m_k = 2\pi\) and (5) becomes:
\[
\sum_{k=1}^{n} \frac{a_k^2}{x_k x_{k+1}} \geq 4n \sin^2 \frac{\pi}{n},
\]
i.e. the problem 11386, proposed by R.N. Gologan in G.M.-B nr. 8/1971, pp.487.

**REFERENCES**