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MODULES THAT HAVE A WEAK δ-SUPPLEMENT IN EVERY TORSION EXTENSION

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Abstract. We study modules with the properties (δ – TWE) and (δ – TWEE) which are adopted Zöschinger’s modules with the properties (E) and (EE). We call a module (δ – TWE) module if M has a weak δ-supplement in every torsion extension. Similarly if M has ample weak δ-supplements in every torsion extension then M is called (δ – TWEE) module. We obtain various properties of these modules. We will show that (1) Every direct summand of a (δ – TWE) module is a (δ – TWE) module. (2) A module M has the property (δ – TWEE) iff every submodule of M has the property (δ – TWE). (3) Any factor module of a (δ – TWE) module is a (δ – TWE) module under a special condition. (4) Over a non-local ring, if every submodule of a module M is a (δ – TWE) module, then it is cofinitely weak δ-supplemented.

Keywords: δ-small submodule, weak δ-supplement, torsion extension.

1. INTRODUCTION

Throughout this paper R will be a commutative domain and all modules are unital left R-modules unless otherwise stated. Let M be an R-module. By N ≤ M we mean that N is a submodule of M. Recall that a submodule N of M is called small, denoted by N ≪ M, if N + L ≠ M for all proper submodules L of M. Dually a submodule L of M is said to be essential in M, denoted by L ⊳ M, if L ∩ K ≠ 0 for each non zero submodule K of M. [13] A module M is said to be singular if M ≅ N/L for some module N and a submodule L of N with L ⊳ N [5].

As a generalization of direct summands of a module one can define supplement submodules. A module M is called supplemented, if every submodule N of M has a supplement in M, i.e. a submodule K of M minimal with respect to M = N + K. Equally, K is a supplement of N in M iff M = N + K and N ∩ K ≪ K. If N + K = M and N ∩ K ≪ M, then K is called a weak supplement of N in M. M is weakly supplemented module if every submodule of M has a weak supplement in M. A submodule N of a module M has ample (weak) supplements in M if for all K ⊳ M with M = N + K, there is a (weak) supplement K′ of N with K′ ≤ K. If every submodule of M has ample (weak) supplements in M, then M is called amply (weak) supplemented [13].

The concept of δ-small submodules was introduced by Zhou in [14], as a generalization of small submodules. A submodule N ≤ M is said to be δ-small in M if N + X ≠ M for all proper X ⊳ M with M/X singular. The sum of all δ-small submodules of a module M is denoted by δ(M). Let K, N be submodules of a module M. N is called a δ-
supplement of \( K \) in \( M \), if \( M = N + K \) and \( N \cap K \ll_\delta N \) [7]. Similarly \( N \) is called a weak \( \delta \)-supplement of \( K \) in \( M \), if \( M = N + K \) and \( N \cap K \ll_\delta M \) [11]. A module \( M \) is called (weak) \( \delta \)-supplemented if every submodule of \( M \) has a (weak) \( \delta \)-supplement in \( M \). On the other hand, a submodule \( N \) of \( M \) is said to have ample (weak) \( \delta \)-supplements in \( M \) if every submodule \( L \) of \( M \) with \( M = N + L \) contains a (weak) \( \delta \)-supplement of \( N \) in \( M \). The module \( M \) is called amply (weak) \( \delta \)-supplemented if every submodule of \( M \) has ample (weak) \( \delta \)-supplement in \( M \) [11, 12].

Let \( R \) be a commutative domain and \( M \) be an \( R \)-module. We denote by \( T(M) \) the set of all elements \( m \) of \( M \) for which there exists a nonzero element \( r \) of \( R \) such that \( rm = 0 \) i.e. \( Ann(m) \neq 0 \). Then \( T(M) \), which is a submodule of \( M \), called the torsion submodule of \( M \). Especially \( M \) is called torsion module provided that \( T(M) = M \) [13].

For modules \( M \leq N \) over commutative domain, we say that \( N \) is a torsion extension of \( M \) if \( N/M \) is torsion. Göçer and Türkmen in [6], studied modules with the property \((\delta-TE)\) i.e. modules that have a supplement in every torsion extension. Eryilmaz in [4], studied modules with the property \((\delta-TE)\). Motivated by these we introduce \((\delta-TWE)\) modules i.e. modules that have a weak \( \delta \)-supplement in every torsion extension. In this study we obtain various properties of modules with the property \((\delta-TWE)\). We show that a class of \((\delta-TWE)\) modules is closed under direct summands and factor modules by a special condition. We prove that every submodule of a module is a \((\delta-TWE)\) module iff it has ample weak \( \delta \)-supplements in every torsion extension. We also show that over a non local ring if every submodule of a module \( M \) is a \((\delta-TWE)\) module then it is cofinitely weak \( \delta \)-supplemented.

2. PRELIMINARIES

We will give following lemmas for the completeness.

**Lemma 1:** Let \( M \) be an \( R \)-module, then the following statements are equivalent:
1. \( M \) is cofinitely \( \delta \)-supplemented
2. Every maximal submodule of \( M \) has a \( \delta \)-supplement in \( M \) [1].

**Lemma 2:** Let \( R \) be a ring which is not local. If \( M \) is a simple module then it is torsion [6].

3. RESULTS AND DISCUSSION

**Proposition 1:** \( \delta \)-Hollow modules have the property \((\delta-TWE)\).

**Proof:** Let \( S \) be a \( \delta \)-hollow module and \( N \) be any torsion extension of \( S \). If \( S \) is \( \delta \)-small in \( N \), \( N \) is a weak \( \delta \)-supplement of \( S \) in \( N \). Suppose that \( S \) is not \( \delta \)-small in \( M \). Then there is a proper submodule \( S' \) of \( N \) such that \( S + S' = N \) and \( N/S \) is singular. If \( S \) is simple \( S \cap S' = 0 \) and so \( S' \) is a direct summand of \( N \). In opposite situation since \( S \) is \( \delta \)-hollow, \( S \cap S' \) is \( \delta \)-small in \( S \). In both cases, \( S' \) is a weak \( \delta \)-supplement of \( S \) in \( N \).

**Proposition 2:** Every direct summand of a \((\delta-TWE)\) module is a \((\delta-TWE)\) module.
Proof: Let $M$ be a $(\delta - TWE)$ module, $U$ be a direct summand of $M$ and let $N$ be any torsion extension of $U$. Then $M = A \oplus U$ for some submodule $A \leq M$. We denote by $T$ the external direct sum $A \oplus N$ and consider the canonical embedding $\varphi: M \to T$. Then $M \cong \varphi(M)$ is a $(\delta - TWE)$ module and we have $T/\varphi(M) = (A \oplus N)/\varphi(M) \cong (A \oplus N)/(A \oplus U) \cong N/U$ is torsion. Since $\varphi(M)$ is a $(\delta - TWE)$ module, $\varphi(M)$ has a weak $\delta$-supplement $V$ in $T$, that is, $\varphi(M) + V = T$ and $\varphi(M) \cap V \ll \delta T$. For the projection $\pi: T \to N$, we have that $N = U + \pi(V)$. Since $\text{Ker}(\pi) \subseteq \varphi(M)$, we get $\pi(\varphi(M) \cap V) \subseteq \pi(\varphi(M)) \cap \pi(V) = U \cap \pi(V) \ll \delta \pi(T) \leq N$ and so $U \cap \pi(V) \ll \delta N$ is obtained. Hence $\pi(V)$ is a weak $\delta$-supplement of $U$ in $N$.

Proposition 3: Let $M$ be a module. Then the following statements are equivalent:

1. Every submodule of $M$ is a $(\delta - TWE)$ module.
2. $M$ has ample $\delta$-supplements in every torsion extension i.e. $M$ is a $(\delta - TWEE)$ module.

Proof: (1) $\Rightarrow$ (2): Suppose that every submodule of $M$ is a $(\delta - TWE)$ module. For a torsion extension $N$ of $M$, let $N = M + K$ for some submodule $K$ of $N$. Note that $N/M = (M + K)/M \cong K/(M \cap K)$ is torsion. By hypothesis $M \cap K$ is a $(\delta - TWE)$ module and so there exists a submodule $L$ of $K$ such that $K = (M \cap K) + L$ and $(M \cap K) \cap L = M \cap L \ll \delta K$. Then we have $M \cap L \ll \delta N$ and $N = M + K = M + (M \cap K) + L = M + L$. Hence $L$ is a weak $\delta$-supplement of $M$ in $N$.

(2) $\Rightarrow$ (1): Let $M$ be a module with the property $(\delta - TWEE)$ and let $U$ be any submodule of $M$. For a cofinite extension $N$ of $U$, let $F = (M \oplus N)/H$ where the submodule $H$ is the set of all elements $(a, -a)$ of $F$ with $a \in U$ and let $\alpha: M \to F$ via $\alpha(m) = (m, 0) + H$, $\beta: N \to F$ via $\beta(n) = (0, n) + H$ for all $m \in M$, $n \in N$. It is clear that $\alpha$ and $\beta$ are monomorphisms. Hence we have the following pushout diagram:

where $\mu_1: U \to N$ and $\mu_2: U \to M$ are inclusion mappings. It is easy to prove that $F = \text{Im}(\alpha) + \text{Im}(\beta)$. Now we define $\gamma: F \to N/U$ by $\gamma((m, n) + H) = n + U$ for all $(m, n) + H \in F$. Then $\gamma$ is an epimorphism. Note that $\text{Ker}(\gamma) = \text{Im}(\alpha)$ and so $N/U \cong F/\text{Im}(\alpha)$ is finitely generated. Since $\alpha$ is a monomorphism, by assumption, $\text{Im}(\alpha)$ has the property $(\delta - TWEE)$. Then it follows immediately that $\text{Im}(\alpha)$ has a weak $\delta$-supplement $V$ in $F$ with $V \leq \text{Im}(\beta)$, i.e. $F = \text{Im}(\alpha) + V$ and $\text{Im}(\alpha) \cap V \ll \delta F$. Then $N = \beta^{-1}(\text{Im}(\alpha)) + \beta^{-1}(V) = U + \beta^{-1}(V)$. Suppose that $U \cap \beta^{-1}(V) + X = N$ for some submodule $X$ of $N$ with $N/X$ singular. Then we have $\beta((U \cap \beta^{-1}(V) + X) = \beta(U \cap \beta^{-1}(V)) + \beta(X) = \text{Im}(\alpha) \cap V + \beta(X) = \beta(N)$ since $\beta$ is a monomorphism. And it is clear that $\text{Im}(\alpha) \cap V + \beta(X) = \text{Im}(\alpha) + \text{Im}(\beta)$ is singular. Then we have $\beta(U \cap \beta^{-1}(V) + X) = \beta(U \cap \beta^{-1}(V)) + \beta(X) = \text{Im}(\alpha) \cap V + \beta(X) = \beta(N)$ is obtained because of definition of $\beta$. And we have that $X = N$ since $\beta$ is a
monomorphism. That means $U \cap \beta^{-1}(V) \ll_{\delta} N$. So $\beta^{-1}(V)$ is a weak $\delta$-supplement of $U$ in $N$.

**Proposition 4:** Let $R$ be a ring which is not local and let $M$ be an $R$-module. If every submodule of $M$ is a $(\delta - TE)$ module, then it is cofinitely $\delta$-supplemented.

**Proof:** By [1, Theo. 2.9], it is sufficient to show that every maximal submodule of $M$ has a $\delta$-supplement in $M$. Let $U$ be any maximal submodule of $M$. Then, $M/U$ is simple, and so it is torsion by Lemma 1. By the hypothesis $U$ has a $\delta$-supplement in $M$. Thus $M$ is cofinitely $\delta$-supplemented.

**Definition 5:** We call a module $M$ is cofinitely weak $\delta$-supplemented module (or briefly $\delta$-cws module) if every cofinite submodule has a weak $\delta$-supplement in $M$.

Clearly cofinitely $\delta$-supplemented modules and weakly $\delta$-supplemented modules are cofinitely weak $\delta$-supplemented and a finitely generated module is weakly $\delta$-supplemented if and only if it is a $\delta$-cws module.

**Lemma 6:** Let $U$ and $K$ be submodules of $N$ such that $K$ is a weak $\delta$-supplement of a maximal submodule $M$ of $N$. If $K + U$ has a weak $\delta$-supplement in $N$, then $U$ has a weak $\delta$-supplement in $N$.

**Proof:** Let $X$ be a weak $\delta$-supplement of $K + U$ in $N$. If $K \cap (X + U) \leq K \cap M \ll_{\delta} N$ then $X + K$ is a weak $\delta$-supplement of $U$ since $U \cap (X + K) \leq X \cap (K + U) + K \cap (X + U) \ll_{\delta} N$. Now suppose that $K \cap (X + U) \not\ll K \cap M$. Since $K/(K \cap M) \cong (K + M)/M = N/M$, $K \cap M$ is a maximal submodule of $K$. Therefore $(K \cap M) + [K \cap (X + U)] = K$. Then $X$ is a weak $\delta$-supplement of $U$ in $N$ since $U \cap X \leq (K + U) \cap X \ll_{\delta} N$ and $N = X + U + K = X + U + (K \cap M) + [K \cap (X + U)] = X + U$ as $K \cap (X + U) \leq X + U$ and $K \cap M \ll_{\delta} N$. So in both cases there is a weak $\delta$-supplement of $U$ in $N$.

For a module $N$, let $\Gamma$ be the set of all submodules $K$ such that $K$ is a weak $\delta$-supplement for some maximal submodule of $N$ and let $\delta$-cws$(N)$ denote the sum of all submodules from $\Gamma$.

**Theorem 7:** For a module $N$, the following statements are equivalent:

1. $N$ is a $\delta$-cws module;
2. Every maximal submodule of $N$ has a weak $\delta$-supplement;
3. $N/\delta$-cws$(N)$ has no maximal submodules.

**Proof:** (1) $\Rightarrow$ (2) is obvious since every maximal submodule is cofinite.

(2) $\Rightarrow$ (3) Suppose that there is a maximal submodule $M/\delta$-cws$(N)$ of $N/\delta$-cws$(N)$. Then $M$ is a maximal submodule of $N$. By (2), there is a weak $\delta$-supplement $K$ of $M$ in $N$. Then $K \in \Gamma$, therefore $K \leq \delta$-cws$(N) \leq M$. Hence $N = M + K = M$. This contradiction shows that $N/\delta$-cws$(N)$ has no maximal submodules.

(2) $\Rightarrow$ (3) Let $U$ be a cofinite submodule of $N$. Then $U + \delta$-cws$(N)$ is also cofinite. If $N/[U + \delta$-cws$(N)] \neq 0$, by Theorem 2.8 of Anderson and Fuller (1992), there is a maximal submodule $M/[U + \delta$-cws$(N)]$ of the finitely generated module $N/[U + \delta$-cws$(N)]$. It follows that $M$ is a maximal submodule of $N$ and $M/\delta$-cws$(N)$ is a maximal submodule of $N/\delta$-cws$(N)$. This contradicts (3). So $N = U + \delta$-cws$(N)$. Now, $N/U$ is finitely generated, say by elements $x_1 + U, x_2 + U, \ldots, x_m + U$ therefore $N = U + Rx_1 + Rx_2 + \cdots + Rx_m$. Each element $x_i$ ($i = 1, 2, \ldots, m$) can be written as $x_i = u_i + c_i$, where
$u_i \in U, c_i \in \delta$-cws$(N)$. Since each $c_i$ is contained in the sum of finite number of submodules from $\Gamma$, $N = U + K_1 + K_2 + \cdots + K_n$ for some submodules $K_1, K_2, \ldots, K_n$ of $N$ from $\Gamma$. Now $N = (U + K_1 + K_2 + \cdots + K_{n-1}) + K_n$ has a weak $\delta$-supplement, namely 0. By Lemma 6 $U + K_1 + K_2 + \cdots + K_{n-1}$ has a weak $\delta$-supplement. Continuing in this way (applying Lemma 6 n-times) we obtain that $U$ has a weak $\delta$-supplement in $N$.

**Lemma 8:** Let $R$ be a ring which is not local and let $M$ be an $R$-module. If every submodule of $M$ is $(\delta - TWE)$ module then it is cofinitely weak $\delta$-supplemented.

**Proof:** It sufficies to show that every maximal submodule of $M$ has a weak $\delta$-supplement in $M$. Let $U$ be any maximal submodule of $M$. Then $M/U$ is simple and so it is torsion by Lemma 5. By the hypothesis $U$ has a weak $\delta$-supplement in $M$. Thus $M$ is cofinitely weak $\delta$-supplemented.

**Proposition 9:** Let $M$ be a module and $U$ be a submodule of $M$. If $U \ll_{\delta} M$ and the factor module $M/U$ has the property $(\delta - TWE)$, then $M$ also has the property $(\delta - TWE)$.

**Proof:** Let $N$ be any torsion extension of $M$. Then we obtain that $N/M \cong (N/U)/(M/U)$ is torsion. Since $M/U$ has the property $(\delta - TWE)$, there exists a submodule $V/U$ of $N/U$ such that $M/U + V/U = N/U$ and $(M/U) \cap (V/U) = (M \cap V)/U \ll_{\delta} N/U$. Note that $M + V = N$. Suppose that $(M \cap V) + T = N$ for a submodule $T$ of $N$ such that $N/T$ is singular. Then we obtain $(M \cap V)/U + (T + U)/U = N/U$. Since $(M \cap V)/U \ll_{\delta} N/U$ and $N/T + N \leq N/T$ singular we have that $(T + U)/U = N/U$. It is clear that $T + U = N$. By hypothesis and since $N/T$ is singular it follows that $N = T$. Hence $M \cap V \ll_{\delta} N$.

**Corollary 10:** Let $M$ be a finitely generated module. If $M/\delta(M)$ has the property $(\delta - TWE)$, then so does $M$.

**Lemma 11:** Let $M$ be a $(\delta - TWE)$ module and $N$ be a torsion extension of $M$ such that $\delta(N) = 0$. Then $M$ is a direct summand of $N$.

**Proof:** By assumption, $M$ has a weak $\delta$-supplement in $N$. Since $M \cap K \ll_{\delta} K$, it follows from $M \cap V \leq \delta(K) \leq \delta(N) = 0$. Hence $N = M \oplus K$.

**Corollary 12:** Let $M$ be a $(\delta - TWE)$ module over $\delta$-$V$-ring. Then $M$ is a direct summand of any module $N$ with $N/M$ torsion.

**Theorem 13:** Let $A \leq B \leq C$ with $(C/A)$ injective. If $B$ has the property $(\delta - TWE)$, so does $B/A$.

**Proof:** Let $N$ be any extension of $B/A$. So we have the following commutative diagram with exact rows since $C/A$ is injective [by 16, Lemma 2.16].
Since $h$ is monic, $N / (B / A) \cong g(N) / g(B / A) \cong \sigma^{-1}(g(N))/\sigma^{-1}(g(B/A)) = \sigma^{-1}(g(N)) / \sigma^{-1}(\sigma(B)) \cong \sigma^{-1}(g(N)) / B$ is torsion and $B$ has the property $(\delta - TWE)$, $B \cong h(B)$ has a weak $\delta$-supplement $V$ in $P$ (that is, $h(B) + V = P$ and $h(B) \cap V \ll_{\delta} P$. We claim that $g(V)$ is a weak $\delta$-supplement of $B/A$ in $N$.

\[ B/A + g(V) = (f \sigma(B) + g(V) = g(h(B)) + g(V) = g(P) = N, B/A \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap g(V)] \ll_{\delta} g(V) \]

since $h(B) \cap V \ll_{\delta} V$ and $g$ is a homomorphism. Hence $B/A \cap g(V) \ll_{\delta} N$.

**Corollary 14:** Let $R$ be a hereditary ring. If an injective $R$-module $M$ has the property $(\delta - TWE)$, then so does every factor module of $M$.

**REFERENCES**