THE DECOMPOSITION OF SOME ANNIHILATOR POLYNOMIALS FOR LINEAR MAPS IN COPRIME POLYNOMIALS

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Abstract. The decomposition of annihilator polynomials for linear maps or for the matrices of product of two coprime polynomials gives a decomposition of the spectrum as a direct sum of two subspaces with annihilator polynomials of smaller degree.

Keywords: Linear Map, Range and Null space, Rank, Idempotent and Involuntary matrices.


1. INTRODUCTION

Let $V$ be a vector space over a field $K$ and let $T : V \to V$ be a linear map for which there is an annihilator polynomial $P \in K[X]$, i.e. $P(T) = 0$. If the annihilator polynomial $P$ is irreductible then there are two coprime, non-constant polynomials such that $P = P_1 \cdot P_2$. We give some relations in which the null spaces $\ker(P_1(T))$, $\ker(P_2(T))$ and the numbers $\text{rank}(P_1(T))$, $\text{rank}(P_2(T))$ appear.

2. COMPLEMENTARY SUBSPACES $\ker(P_1(T)) \text{ AND } \ker(P_2(T))$

Theorem 2.1 If $V$ is a vector space over the field $K$ with $T : V \to V$ an endomorphism and $P_1, P_2 \in K[X]$ two coprime polynomials then the following statements are equivalent:

a) $P_1(T) \cdot P_2(T) = 0$ or $(P_1 \cdot P_2)(T) = 0$

b) $V = \ker(P_1(T)) \oplus \ker(P_2(T))$.

Proof: Since the polynomials are coprime there are two polynomials $Q_1, Q_2 \in K[X]$ so that

$$P_1Q_1 + P_2Q_2 = 1$$

(1)

a) $\to$ b): In the hypothesis $P_1(T) \cdot P_2(T) = 0$. We show that any vector $x \in V$ can be decomposed uniquely as $x = x_1 + x_2$ with $x_1 \in \ker(P_1(T))$ and $x_2 \in \ker(P_2(T))$. Since $(P \cdot Q)(T) = P(T) \cdot Q(T) = Q(T) \cdot P(T)$, from (1) we obtain $P_1(T) \cdot Q_1(T) + P_2(T) \cdot Q_2(T) = I$, where $I : V \to V$ is the identity map $I(x) = x, x \in V$. We select $x_1 = P_2(T) \cdot Q_2(T)(x)$ and $x_2$ as the unique such that $x_1 + x_2 = x$.

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\[ x_2 = P_1(T) \circ Q_1(T)(x) \]
and we have
\[ P_1(T)(x_2) = (P_1(T) \circ P_2(T))(Q_2(T)(x)) = 0, \]
hence \( x_1 \in \ker(P_1(T)) \) and \( P_2(T)(x_2) = (P_1(T) \circ P_2(T))(Q_1(T)(x)) = 0 \) so that \( x_2 \in \ker(P_2(T)) \).

Moreover, if
\[ y \in \ker(P_1(T)) \cap \ker(P_2(T)) \]
then
\[ P_1(T)(y) = 0, \ P_2(T)(y) = 0 \]
and
\[ y = (P_1 \circ P_2)(T)(y) = Q_1(T)(P_1(T)(y)) + Q_2(T)(P_2(T)(y)) = 0 \]
hence
\[ \ker(P_1(T)) \cap \ker(P_2(T)) = \{0\} \]
and then \( V = \ker(P_1(T)) \oplus \ker(P_2(T)) \).

b) → a): Any vector \( x \in V \) can be written uniquely as \( x = x_1 + x_2 \) with \( x_1 \in \ker(P_1(T)) \), \( x_2 \in \ker(P_2(T)) \). We have,
\[
P_1(T) \circ P_2(T)(x) = P_1(T)(P_2(T)(x_1)) + P_1(T)(P_2(T)(x_2)) = \\
= P_2(T)(P_1(T)(x_1)) + P_1(T)(P_2(T)(x_2)) = P_2(T)(0) + P_1(T)(0) = 0
\]
hence \( P_1(T) \circ P_2(T) = 0 \).

**Corollary 2.1** The linear map \( T : V \to V \) is a linear projection on vector space \( V \) if and only if \( V = \ker(T) \oplus \ker(I - T) \).

**Proof:** The map \( T \) is a linear projection if and only if the polynomial \( P(x) = x - x^2 \) is the annihilator polynomial for \( T \). We consider the polynomials \( P_1(x) = x, \ P_2(x) = 1 - x \) which are coprime and \( P(x) = P_1(x) \cdot P_2(x) \).

We have \( P(T) = 0 \iff (P \circ P_2)(T) = 0 \iff P_1(T) \circ P_2(T) = 0 \) and by Theorem 2.1 this is equivalent to \( V = \ker(P_1(T)) \oplus \ker(P_2(T)) = \ker(T) \oplus \ker(I - T) \).

**Corollary 2.2** If \( V \) is a vector space over the field \( K \) of characteristic different than 2 then the linear map \( S : V \to V \) is a symmetry if and only if \( V = \ker(I - S) \oplus \ker(I + S) \).

**Proof:** The map \( S \) is a symmetry if and only if the polynomial \( P(x) = 1 - x^2 \) is the annihilator polynomial of \( S \ (P(S) = 0) \). We consider the polynomials \( P_1(x) = 1 - x \) and \( P_2(x) = 1 + x \), which in the ring \( K[X] \) are coprime if the characteristic of the field \( K \) is different than 2. We have \( P(x) = P_1(x)P_2(x) \) so that \( 0 = P(S) = P_1(S) \circ P_2(S) \) and by Theorem 2.1 this is equivalent with \( V = \ker(P_1(S)) \oplus \ker(P_2(S)) = \ker(I - S) \oplus \ker(I + S) \).
3. **THE SUM OF THE RANKS** \( \text{rank}(P_1(T)) + \text{rank}(P_2(T)) \)

**Theorem 3.1** Let \( V \) be a vector space of finite dimension over the field \( K \), \( T : V \to V \) an endomorphism and \( P_1, P_2 \in K[X] \) two coprime polynomials. The following statements are equivalent:

a) \( P_1(T) \circ P_2(T) = 0 \)

b) \( \text{rank}(P_1(T)) + \text{rank}(P_2(T)) = \dim(V) \).

**Proof:** a) \( \to \) b): By Theorem 2.1 it follows:

\[
V = \ker(P_1(T)) \oplus \ker(P_2(T)) = \dim(V) - \text{rank}(P_1(T)) + \dim(V) - \text{rank}(P_2(T)),
\]

which implies that \( \text{rank}(P_1(T)) + \text{rank}(P_2(T)) = \dim(V) \).

b) \( \to \) a): From the rank-nullity theorem (see [1]) it follows that

\[
\dim(V) = \dim(\ker(P_1(T))) + \dim(\ker(P_2(T))).
\]

On the other hand, \( \ker(P_1(T)) \cap \ker(P_2(T)) = \{0\} \); since \( P_1(T)(y) = P_2(T)(y) = 0 \) it follows that \( y = (Q, P_1(T))(y) + (Q, P_2(T))(y) = 0 \) hence \( V = \ker(P_1(T)) \oplus \ker(P_2(T)) \) and by Theorem 2.1 we have that \( P_1(T) \circ P_2(T) = 0 \).

**Corollary 3.2** [6] The linear map \( T : V \to V \) is a projection of the vector space \( V \) if and only if \( \text{rank}(T) + \text{rank}(I - T) = \dim(V) \).

**Corollary 3.2** [5] If the field \( K \) has the characteristic different than 2 then the linear map \( S : V \to V \) is a symmetry if and only if \( \text{rank}(I - S) + \text{rank}(I + S) = \dim(V) \).

4. **THE DECOMPOSITION OF ANNIHILATORS POLYNOMIALS FOR MATRICES**

**Theorem 4.1** If \( A \in M_n(K) \) is a matrix and \( P_1, P_2 \in K[X] \) are two coprime polynomials the following statements are equivalent:

a) \( P_1(A) \cdot P_2(A) = 0 \)

b) \( \text{rank}(P_1(A)) + \text{rank}(P_2(A)) = n \).

**Proof:** We consider the linear map \( T : K^n \to K^n \), \( T(X) = AX \), where

\[
X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in K^n
\]

and we have \( P_1(T)(X) = P_1(A)X \), \( P_2(T)(X) = P_2(A)X \), \( \text{rank}(P_1(T)) = \text{rank}(P_1(A)) \), \( \text{rank}(P_2(T)) = \text{rank}(P_2(A)) \) and then Theorem 4.1 is a consequence of Theorem 3.1.
Corollary 4.1 [2, 5] The matrix \( A \in \mathbb{M}_n(K) \) is an idempotent matrix \( (A^2 = A) \) if and only if 
\[ \text{rank}(A) + \text{rank}(I_n - A) = n. \]

Corollary 4.2 [2, 4] If the field \( K \) has \( \text{char}(K) \neq 2 \) then the matrix \( A \in \mathbb{M}_n(K) \) is an 
involutory matrix \( (A^2 = I_n) \) if and only if 
\[ \text{rank}(I_n - A) + \text{rank}(I_n + A) = n. \]

We mention that a recent book which contains a novel presentation of idempotent and 
involutory matrices of order two as well as properties and applications of symmetries and 
projections is [3].

REFERENCES

[3] Pop, V., Furdui, O., Square Matrices of Order Two. Theory Applications and Problems, 
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