MODULES THAT HAVE A WEAK RAD-SUPPLEMENT IN EVERY EXTENSION

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\begin{abstract}
As a proper generalization of the modules with the properties (E) and (EE) that were introduced by Zöschinger in terms of supplements, we say that a module \(M\) has the property \((WRE)\) (respectively, \((WREE)\)) if \(M\) has a weak Rad-supplement (respectively, ample weak Rad-supplements) in every extension. In this paper, we prove that if every submodule of a module \(M\) has the property \((WRE)\), then \(M\) has the property \((WREE)\). We show that a ring \(R\) is semilocal if and only if every left \(R\)-module has the property \((WRE)\). Also we prove that over a commutative Von Neumann regular ring a module \(M\) has the property \((WRE)\) if and only if \(M\) is injective.

\textbf{Keywords:} weak Rad-supplement; extension; semilocal ring; Von Neumann regular ring.

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\end{abstract}

1. INTRODUCTION

Throughout this paper, \(R\) is an associative ring with identity and all modules are unital left \(R\)-modules. Let \(M\) be an \(R\)-module. By \(U \leq M\), we mean that \(U\) is a submodule of \(M\). A submodule \(U\) of \(M\) is said to be \textit{small} in \(M\), denoted as \(U \ll M\), if \(M \neq U + L\) for every proper submodule \(L\) of \(M\). By \(\text{Rad}(M)\), we denote the intersection of all maximal submodules of \(M\) or, equivalently the sum of all small submodules of \(M\). A module \(M\) is called \textit{radical} if \(M\) has no maximal submodules, that is, \(M = \text{Rad}(M)\).

As a proper generalization of direct summands of a module, the notion of supplement submodules is defined. For \(U, V\) submodules of a module \(M\), \(V\) is called a \textit{supplement} of \(U\) in \(M\) if it is minimal with respect to \(M = U + V\), equivalently \(M = U + V\) and \(U \cap V \ll V\). Then, it is natural to introduce a generalization of supplement submodules by [11, 19.3.(2)]. A submodule \(V\) of \(M\) is called a \textit{weak supplement} of \(U\) in \(M\) if \(M = U + V\) and \(U \cap V \ll M\). A submodule \(U\) of \(M\) has \textit{ample} (weak) \textit{supplements} in \(M\) if, whenever \(M = U + L\), \(L\) contains a (weak) supplement of \(U\) in \(M\). A module \(M\) is called \textit{weakly supplemented} if every submodule of \(M\) has a weak supplement in \(M\) (see [5]). By [4, 17.9(6)], if a module \(M\) is weakly supplemented, every submodule of \(M\) has ample weak supplements in \(M\). A submodule \(V\) of \(M\) is called \textit{radical} \textit{supplement} or briefly \textit{Rad-supplement} (according to [10], \textit{generalized} \textit{supplement}) of \(U\) in \(M\) if \(M = U + V\) and \(U \cap V \leq \text{Rad}(V)\) (see [4, 10.14]). A submodule \(U\) of \(M\) has \textit{ample} \textit{Rad-supplements} in \(M\) if every submodule \(L\) of \(M\) with \(M = U + L\) contains a \textit{Rad}-supplement of \(U\) in \(M\). A submodule \(V\) of \(M\) is called \textit{weak} \textit{Rad-supplement} of a

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A submodule $U$ in $M$ if $M = U + V$ and $U \cap V \leq \text{Rad}(M)$. A submodule $U$ of $M$ has ample weak Rad-supplements in $M$ if every submodule $L$ of $M$ with $M = U + L$ contains a weak Rad-supplement of $U$ in $M$. A module $M$ is called weakly Rad-supplemented (according to [10], weakly generalized supplemented) if every submodule of $M$ has a weak Rad-supplement in $M$ [6]. Note that every submodule of a weakly Rad-supplemented module $M$ has ample weak Rad-supplements in $M$. In the view of given definitions, we clearly have the following implication on submodules:

$$
\text{direct summand } \Rightarrow \text{supplement } \Rightarrow \text{Rad – supplement } \Rightarrow \text{weak Rad – supplement}
$$

Let $R$ be a ring and $M$ be an $R$-module. An $R$-module $N$ is called an extension of $M$ provided $M \subseteq N$.

It is well known that a module $M$ is injective if and only if it is a direct summand in every extension. H. Zöschinger initiated the study of the modules that have a supplement (resp. ample supplements) in every extension, i.e. modules with the property $(E)$ (resp. $(EE)$) in [12], as a generalization of injective modules. The author determined in the same paper the structure of modules with these properties.

In [9], S. Özdemir defined the modules that have (ample) a Rad-supplement in every extension, namely (ample) Rad-supplementing modules. He gave various properties of these modules. He provided that every left $R$-module is (ample) Rad-supplementing if and only if $R/P(R)$ is left perfect, where $P(R)$ is the sum of all left ideals $I$ of $R$ such that $\text{Rad}(I) = 1$.

In [8], E. Önal, H. Çalışıcı and E. Türkmen called a module $M$ has the property $(WRE)$ (resp. $(WREE)$) if $M$ has a weak supplement (resp. ample weak supplements) in every extension. Also they gave new characterizations of left perfect rings via the modules that have the property $(WE)$.

Adapting the modules whose definitions were given above, we define and study the modules that have the property $(WRE)$ (resp. $(WREE)$) as a proper generalization of the modules with the property $(E)$ (resp. $(EE)$). A module $M$ is called to have the property $(WRE)$ (resp. $(WREE)$) if it has a weak Rad-supplement (resp. ample weak Rad-supplements) in every extension.

In this paper, we provide some properties of the modules with the properties $(WRE)$ and $(WREE)$. We prove that if every submodule of a module $M$ has the property $(WRE)$, then $M$ has the property $(WREE)$. We show that every direct summand of a module with the property $(WRE)$ has the property $(WRE)$. A ring $R$ is semilocal if and only if every left $R$-module has the property $(WRE)$.

2. MAIN RESULTS

It is shown in [12, Lemma 1.3(a)] that direct summands of modules with the property $(E)$ have the property $(E)$. Now we give an analogue of this fact for the modules with the property $(WRE)$.

Proposition 1: Every direct summand of a module with the property $(WRE)$ has the property $(WRE)$.

Proof: Let $M_1$ be a direct summand of $M$. Then there exists a submodule $M_2$ of $M$ such that $M = M_1 \oplus M_2$. Let $N$ be an extension of $M_1$. Let $N'$ be the external direct sum $N \oplus M_2$ and
Let $M \to N'$ be the canonical embedding. Then $M \cong \theta(M)$ has the property (WRE). Hence, there exists a submodule $V$ of $N'$ such that $N' = \theta(M) + V$ and $\theta(M) \cap V \leq \text{Rad}(N')$. By the projection $\pi: N' \to N$, we have that $M_1 + \pi(V) = N$. Since $\text{Ker}(\pi) \subseteq \theta(M)$, $\pi(\theta(M) \cap V) = \pi(\theta(M)) \cap \pi(V) = M_1 \cap \pi(V) \leq \text{Rad}(N') = \text{Rad}(N)$ by [11, 19.3]. Hence $\pi(V)$ is a weak Rad-supplement of $M_1$ in $N$.

**Proposition 2:** Let $M$ be a module. If every submodule of $M$ has the property (WRE), then $M$ has the property (WREE).

**Proof:** Suppose that every submodule of $M$ has the property (WRE). For any extension $N$ of $M$, let $N = M + K$ for some submodule $K$ of $N$. Since $M \cap K$ has the property (WRE), there exists a submodule $L$ of $K$ such that $(M \cap K) + L = K$ and $(M \cap K) \cap L = M \cap L \leq \text{Rad}(K)$. Note that $N = M + K = M + (M \cap K) + L = M + L$ and $M \cap L \leq \text{Rad}(N)$ by [11, 19.3]. It follows that $L$ is a weak Rad-supplement of $M$ in $N$.

**Proposition 3:** Let $M$ be a module and $U$ be a radical submodule of $M$. If the factor module $M/U$ has the property (WRE), then $M$ has the property (WRE).

**Proof:** Let $N$ be any extension of $M$. Since $M/U$ has the property (WRE), there exists a submodule $V$ of $N/U$ such that $M/U + V = N/U$ and $(M \cap V)/U \leq \text{Rad}(N/U)$. Since $U$ is a radical submodule of $M$, it follows that $U \leq P$ for every maximal submodule $P$ of $M$ and so $\text{Rad}(N/U) = \text{Rad}(N)/U$. Note that $N = M + V$ and $M \cap V \leq \text{Rad}(N)$. Hence $V$ is a weak Rad-supplement of $M$ in $N$.

**Proposition 4:** Every radical module has the property (WRE).

**Proof:** Let $M$ be a radical module and $N$ be any extension of $M$. Since $\text{Rad}(M) = M$, then $N = M + N$ and $M \cap N = M = \text{Rad}(M) \leq \text{Rad}(N)$. Then $N$ is a weak Rad-supplement of $M$ in $N$.

**Corollary 5:** Every finite direct sum of radical modules has the property (WRE).

**Proof:** By [11, 21.6(5)], every finite direct sum of radical modules is a radical module and so it has the property (WRE) by Proposition 4.

Let $P(M) = \sum(U \leq M | \text{Rad}(U) = U)$. A module $M$ is reduced if $P(M) = 0$. Since $P(M)$ is a radical submodule of $M$, we obtain the next result which is a direct consequence of Proposition 4.

**Corollary 6:** For a module $M$, $P(M)$ has the property (WRE).

**Proposition 7:** Let $0 \to K \to M \to L \to 0$ be a short exact sequence and $K$ be a radical module. If $L$ has the property (WRE), $M$ also has the property (WRE). If the sequence splits, the converse holds.
\textbf{Proof}: Without restriction of generality we will assume that $K \leq M$. Since $M/K \cong L$ has the property \((WRE)\) and $K$ is radical, then we have $M$ has the property \((WRE)\) by Proposition 3. On the other hand, suppose that the sequence splits. Then $M \cong K \oplus L$. If $M$ has the property \((WRE)\), then $L$ also has the property \((WRE)\) by Proposition 1.

A ring $R$ is called a \textit{left $V$-ring} if every simple left $R$-module is injective. It is known that $R$ is a left $V$-ring if and only if for any left $R$-module $M$, $\text{Rad}(M) = 0$.

**Proposition 8:** Let $R$ be a left $V$-ring and $M$ be an $R$-module. The following statements are equivalent:

1. $M$ has the property \((WRE)\),
2. $M$ is injective,
3. $M$ has the property \((WE)\).

**Proof:** (1) $\Rightarrow$ (2) Let $N$ be any extension of $M$. Then there exists a weak $\text{Rad}$-supplement $V$ of $M$ in $N$, that is, $M + V = N$, $M \cap V \leq \text{Rad}(N)$. Since $R$ is left $V$-ring, $\text{Rad}(N) = 0$. Hence we have that $N = M \oplus V$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear.

A ring $R$ is called \textit{Von Neumann regular} if every element $a$ of $R$ can be written in the form $axa$ for some $x \in R$. A commutative ring is a left $V$-ring if and only if it is Von Neumann regular [11, 23.5]. The next result is a direct consequence of this fact and Proposition 8.

**Corollary 9:** Let $R$ be a commutative Von Neumann regular ring. Then an $R$-module $M$ is injective if and only if it has the property \((WRE)\).

In general, a module with the property \((WRE)\) does not need to have the property \((WE)\). To see this, we shall consider the following example:

**Example 10:** (see [2, Example 6.2]) Let $k$ be a field. In the polynomial ring $k[x_1, x_2, \ldots]$ with countably many indeterminates $x_n$, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by $x_1^2$ and $x_{n+1}^2 - x_n$ for every $n \in \mathbb{Z}^+$. Then the quotient ring $R = \frac{k[x_1, x_2, \ldots]}{I}$ is a local ring and the ideal $J = (x_1, x_2, \ldots)$ of $R$ generated by all $x_n + I$, $n \in \mathbb{Z}^+$ is the unique maximal ideal of $R$. Then $M = J^{(N)}$ is a radical $R$-module and so $M$ has the property \((WRE)\) by Proposition 4. Since $R$ is not a left perfect ring, $M = J^{(N)} = \text{Rad}(R^{(N)})$ does not have a weak supplement in $R^{(N)}$ by [1, Theorem 1]. Thus $M$ does not have the property \((WE)\).

We recall from [4] that a ring $R$ is a \textit{left Bass ring} if and only if for every nonzero left $R$-module $M$, $\text{Rad}(M) \ll M$.

**Lemma 11:** Let $R$ be a left Bass ring and $M$ be an $R$-module. Then every weak $\text{Rad}$-supplement in $M$ is a weak supplement in $M$. 

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Proof: Let \( V \) be a weak \( \text{Rad} \)-supplement submodule in \( M \). Then there exists a submodule \( U \) of \( M \) such that \( M = U + V \) and \( U \cap V \subseteq \text{Rad}(M) \). Since \( R \) is a left Bass ring, \( \text{Rad}(M) \ll M \). Hence \( V \) is a weak supplement of \( U \) in \( M \).

\[
\text{Theorem 12:} \text{ Let } R \text{ be a left Bass ring. Then the followings are equivalent:}
\]

1. Every left \( R \)-module has the property \( (WRE) \).
2. Every left \( R \)-module has the property \( (WE) \).

Proof: \( (1) \Rightarrow (2) \) Let \( M \) be an \( R \)-module with the property \( (WRE) \), and \( N \) be any extension of \( M \). By hypothesis, there exists a weak \( \text{Rad} \)-supplement \( V \) of \( M \) in \( N \). \( V \) is a weak supplement of \( M \) in \( N \) by Lemma 11. Thus \( M \) has the property \( (WE) \).

\( (2) \Rightarrow (1) \) is clear.

Recall from [3] that a module \( M \) is called strongly radical supplemented if every submodule of \( M \) containing \( \text{Rad}(M) \) has a supplement in \( M \). It is shown in [3, Corollary 2.1] that finite sums of strongly radical supplemented modules are strongly radical supplemented. Moreover every radical module is strongly radical supplemented by [3, Lemma 2.2]. In general, modules with the property \( (WRE) \) need not to have the property \( (E) \) as the following example shows.

\[
\text{Example 13:} \text{ (see [7, Example 1]) For a non-complete local dedekind domain } R, \text{ let } M = \text{ the direct sum of left } R\text{-modules } R^*, K^{(l)} \text{ and } R, \text{ where } R^* \text{ is the completion of } R, K \text{ is the quotient field of } R \text{ and } I \text{ is an index set, respectively. Since injective modules over a dedekind domain are strongly radical supplemented, it follows from [12, Lemma 3.3] that } M \text{ has the property } \text{(WRE)}. \text{ On the other hand, } M \text{ doesn't have the property } (E) \text{ by [12, Theorem 3.5].}
\]

Next we prove that a factor module of a module with the property \( (WRE) \) has the property \( (WRE) \), under a special condition.

\[
\text{Proposition 14:} \text{ Let } K \subseteq M \subseteq L \text{ be modules with } L/K \text{ injective. If } M \text{ has the property } \text{(WRE)}, \text{ then } M/K \text{ also has the property } \text{(WRE)}. \]

Proof: Let \( N \) be any extension of \( M/K \). Since \( L/K \) is injective, by [9, Lemma 2.16] we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\
& & \downarrow{id} & & \downarrow{h} & & \downarrow{f} & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0
\end{array}
\]

Since \( h \) is monomorphism and \( M \) has the property \( (WRE) \), \( M \cong h(M) \) has a weak \( \text{Rad} \)-supplement \( V \) in \( P \), that is, \( h(M) + V = P \) and \( h(M) \cap V \subseteq \text{Rad}(P) \). Note that \( N = g(P) = g(h(M)) + g(V) = (f \sigma)(M) + g(V) = M/K + g(V) \) and by [11, 21.6], \( M/K \cap g(V) = f(\sigma(M)) \cap g(V) = g[h(M) \cap V] \leq g[\text{Rad}(P)] = \text{Rad}(N) \). Hence \( g(V) \) is a weak \( \text{Rad} \)-supplement of \( M/K \) in \( N \), that is, \( M/K \) has the property \( (WRE) \).

A ring \( R \) is called left hereditary if every left ideal of \( R \) is projective in \( _RR \). It is well
known that a ring $R$ is left hereditary if and only if every factor module of an injective $R$-module is injective [11, 39.16].

**Corollary 15:** If $R$ is a left hereditary ring and $M$ is an $R$-module with the property ($\text{WRE}$), then every factor module of $M$ has the property ($\text{WRE}$).

We also obtain the following result by Proposition 3 and Proposition 14.

**Corollary 16:** Let $R$ be a left hereditary ring and $M$ be an $R$-module. Then $\frac{M}{M/\mathcal{P}(M)}$ has the property ($\text{WRE}$) if and only if $M$ has the property ($\text{WRE}$).

**Theorem 17:** For a ring $R$ the following statements are equivalent:

1. $R$ is semilocal,
2. Every left $R$-module is weakly $\text{Rad}$-supplemented,
3. Every left $R$-module has the property ($\text{WRE}$),
4. Every left $R$-module has the property ($\text{WREE}$).

**Proof:** (1) $\Rightarrow$ (2) Let $M$ be an $R$-module and $U$ be a submodule of $M$. Since $R$ is semilocal ring, $M$ is a semilocal module by [5, Theorem 3.5]. Then there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \leq \text{Rad}(M)$ by [5, Proposition 2.1]. Hence $M$ is weakly $\text{Rad}$-supplemented.

(2) $\Rightarrow$ (3) Let $M$ be a left $R$-module and $N$ be any extension of $M$. By hypothesis, $N$ is weakly $\text{Rad}$-supplemented, and so $M$ has a weak $\text{Rad}$-supplement in $N$.

(3) $\Rightarrow$ (4) Let $M$ be a left $R$-module. By hypothesis, every submodule of $M$ has the property ($\text{WRE}$). Then $M$ has the property ($\text{WREE}$) by Proposition 2.

(4) $\Rightarrow$ (1) Let $M$ be a left $R$-module with small radical and $U$ be a submodule of $M$. Since $U$ has the property ($\text{WREE}$), there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \leq \text{Rad}(M)$. Then $V$ is a weak supplement of $U$ in $M$, because $\text{Rad}(M) \ll M$. Thus $M$ is weakly supplemented. Hence $R$ is semilocal by [5, Theorem 3.5].

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