Abstract. In this paper, we deal with a $q$-Dirac system. We investigate some spectral properties and the asymptotic behavior of the eigenvalues and the eigenfunctions of this $q$-Dirac system.

Keywords: $q$-Dirac system, eigenvalues and eigenfunctions, eigenfunction expansions

1. INTRODUCTION

We consider a $q$-Dirac system which consists of the system of $q$-differential equations

\[
\begin{align*}
-\frac{1}{q} D_q^{-1} y_2(x) + p(x) y_1(x) &= \lambda y_1(x), \\
D_q y_1(x) + r(x) y_2(x) &= \lambda y_2(x),
\end{align*}
\]

and the boundary conditions

\[
\begin{align*}
B_1(y) := k_{11} y_1(0) + k_{12} y_2(0) &= 0, \\
B_2(y) := k_{21} y_1(a) + k_{22} y_2(a q^{-1}) &= 0,
\end{align*}
\]

where $k_{ij} (i, j = 1, 2)$ are real numbers, $\lambda$ is a complex eigenvalue parameter, $y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$, $0 \leq x \leq a < \infty$, $p(x)$ and $r(x)$ are real-valued functions defined on $[0, a]$ and continuous at zero and $p(x), r(x) \in L_q^1(0, a)$ (see[1]).

In [1], the authors introduced a $q$-analog of one-dimensional Dirac equation (1) and they investigated the existence and uniqueness of the solution of this equation and also gave some spectral properties of the problem (1)-(3). Dissipative, accumulative, self-adjoint for the same $q$-Dirac equation were described in [2].

In this paper, we study similar spectral properties and obtain the asymptotic formulas of the eigenvalues and the eigenfunctions of the problem (1)-(3) in the light of the theory of $q$-(basic) Sturm-Liouville problems [3, 4].

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2. PRELIMINARIES

In this section we introduce some of the required $q$-notations and results. Throughout this paper $q$ is a positive number with $0 < q < 1$.

A set $A \subseteq \mathbb{R}$ is called $q$-geometric if, for every $x \in A$, $qx \in A$. Let $f$ be a real or complex-valued function defined on a $q$-geometric set $A$. The $q$-difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x(1-q)}, x \neq 0. \quad (4)$$

If $0 \in A$, the $q$-derivative at zero is defined to be

$$D_q f(0) := \lim_{n \to \infty} \frac{f(x^n) - f(0)}{x^n}, x \in A, \quad (5)$$

if the limit exists and does not depend on $x$. Also, for $x \in A$, $D_q^{-1}$ is defined to be

$$D_q^{-1} f(x) := \begin{cases} \frac{f(x) - f(q^{-1}x)}{x(1-q^{-1})}, & x \in A \setminus \{0\}, \\ D_q f(0), & x = 0, \end{cases} \quad (6)$$

provided that $D_q f(0)$ exists. The following relation can be verified directly from the definition

$$D_q^{-1} y(x) = (D_q y)(xq^{-1}). \quad (7)$$

A right inverse, $q$-integration, of the $q$-difference operator $D_q$ is defined by Jackson [5] as

$$\int_0^x f(t) d_q t := x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n), x \in A, \quad (8)$$

provided that the series converges. A $q$-analog of the fundamental theorem of calculus is given by

$$D_q \int_0^x f(t) d_q t = f(x), \quad \int_0^x D_q f(t) d_q t = f(x) - \lim_{n \to \infty} f(xq^n), \quad (9)$$

where $\lim_{n \to \infty} f(xq^n)$ can be replaced by $f(0)$ if $f$ is $q$-regular at zero, that is, if

$$\lim_{n \to \infty} f(xq^n) = f(0), \text{ for all } x \in A. \quad \text{Throughout this paper, we deal only with functions } q\text{-regular at zero.}$$

The $q$-type product formula is given by
\[ D_q(fg)(x) = g(x)D_qf(x) + f(qx)D_qg(x), \quad (10) \]

and hence the \( q \)-integration by parts is given by

\[ \int_0^a g(x)D_qf(x)d_qx = (fg)(a) - (fg)(0) - \int_0^a D_qg(x)f(qx)d_qx, \quad (11) \]

where \( f \) and \( g \) are \( q \)-regular at zero.

For more results and properties in \( q \)-calculus, readers are referred to the recent works [6-9].

The basic trigonometric functions \( \cos(z;q) \) and \( \sin(z;q) \) are defined on \( \mathbb{C} \) by

\[ \cos(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(q;q)_{2n}} (z(1-q))^{2n}, \quad (12) \]

\[ \sin(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}}{(q;q)_{2n+1}} (z(1-q))^{2n+1}, \quad (13) \]

and they are \( q \)-analogs of the cosine and sine functions [7, 10, 11].

**Theorem 2.1.** ([12]) If \( \{x_m\} \) and \( \{y_m\} \) are the positive zeros of \( \cos(z;q) \) and \( \sin(z;q) \), respectively, then we have for sufficiently large \( m \),

\[ \{x_m\} = q^{-m+1/2} (1-q)^{-1} \left( 1 + O(q^m) \right), \quad (14) \]

\[ \{y_m\} = q^{-m} (1-q)^{-1} \left( 1 + O(q^m) \right). \quad (15) \]

**Corollary 2.1.** ([12, Corollaries 3.2 and 3.4]) For \( r := |z| \to \infty \) we have

\[ M(r;\cos(z;q)) = O \left( \exp \left( -\frac{(\log r (1-q))^2}{\log q} \right) \right), \quad (16) \]

\[ M(r;\sin(z;q)) = O \left( \exp \left( -\frac{(\log r (1-q))^2}{\log q} \right) \right). \quad (17) \]

**3. SPECTRAL PROPERTIES AND ASYMPTOTIC FORMULAS**

In this section we give some spectral properties similar to [1, 13], then we obtain asymptotic formulas for the eigenvalues and the eigenfunctions of the problem (1)-(3).
Lemma 3.1. The eigenfunctions \( y(x, \lambda_1) \) and \( z(x, \lambda_2) \) corresponding to different eigenvalues \( \lambda_1 \neq \lambda_2 \) are orthogonal, i.e.,

\[
\int_0^a y^+ z \, dq \, x = \int_0^a \left\{ y_1(x, \lambda_1) z_1(x, \lambda_2) + y_2(x, \lambda_1) z_2(x, \lambda_2) \right\} \, dq \, x = 0, \quad (18)
\]

where \( y^+ = (y_1, y_2) \), \( \perp \) denotes the matrix transpose.

Proof. Since \( y(x, \lambda_1) \) and \( z(x, \lambda_2) \) are solutions of the \( q \)-system (1),

\[
\begin{aligned}
-\frac{1}{q} D_q y_2 + \left\{ p(x) - \lambda_1 \right\} y_1 &= 0, \\
D_q y_1 + \left\{ r(x) - \lambda_1 \right\} y_2 &= 0, \\
-\frac{1}{q} D_q z_2 + \left\{ p(x) - \lambda_2 \right\} z_1 &= 0, \\
D_q z_1 + \left\{ r(x) - \lambda_2 \right\} z_2 &= 0.
\end{aligned}
\]

Multiplying by \( z_1, z_2, -y_1 \) and \( -y_2 \), respectively, and adding together and also using the formulas (7) and (10), we obtain

\[
D_q \left( y_1(x, \lambda_1) z_2 \left( xq^{-1}, \lambda_2 \right) - y_2 \left( xq^{-1}, \lambda_1 \right) z_1 (x, \lambda_2) \right) = (\lambda_1 - \lambda_2) \left\{ y_1(x, \lambda_1) z_1(x, \lambda_2) + y_2(x, \lambda_1) z_2(x, \lambda_2) \right\}. \quad (19)
\]

Applying the \( q \)-integration to (19), we have

\[
(\lambda_1 - \lambda_2) \int_0^a y^+ (x, \lambda_1) z(x, \lambda_2) \, dq \, x = \left\{ y_1(x, \lambda_1) z_2 \left( xq^{-1}, \lambda_2 \right) - y_2 \left( xq^{-1}, \lambda_1 \right) z_1 (x, \lambda_2) \right\}_0^a. \quad (20)
\]

It follows from the boundary conditions (2) and (3) the right hand side vanishes. Therefore, we get

\[
(\lambda_1 - \lambda_2) \int_0^a y^+ (x, \lambda_1) z(x, \lambda_2) \, dq \, x = 0. \quad (21)
\]

The lemma is thus proved, since \( \lambda_1 \neq \lambda_2 \).

Lemma 3.2. The eigenvalues of the problem (1)-(3) are real.

Proof. Assume the contrary that \( \lambda_0 \) is a nonreal eigenvalue of the problem (1)-(3). Let \( y(x, \lambda_0) \) be a corresponding (nontrivial) eigenfunction. \( \lambda_0 \) is also an eigenvalue, corresponding to the eigenfunction \( \overline{y}(x, \lambda_0) \). Since \( \lambda_0 \neq \lambda_0 \) by the previous lemma,
\[ \int_0^a \left( \left\| y_1(x, \lambda_0) \right\|^2 + \left\| y_2(x, \lambda_0) \right\|^2 \right) \, dq = 0. \]  

(22)

Hence \( y(x, \lambda_0) \equiv 0 \) and this is a contradiction. Consequently, \( \lambda_0 \) must be real. □

Now, we will construct a special system of solution of \( q \)-system (1). Let

\[ \phi(x, \lambda) = \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix} \]

be a solution of the \( q \)-system (1) that satisfies the initial conditions

\[ \phi_1(0, \lambda) = k_{12}, \quad \phi_2(0, \lambda) = -k_{11}. \]  

(23)

The existence and uniqueness of this solution for the \( q \)-system (1) were presented in [1]. It is obvious that \( \phi(x, \lambda) \) satisfies the boundary condition (2).

**Theorem 3.1.** The following integral equations hold for the solution of \( \phi(x, \lambda) \)

\[
\begin{aligned}
\phi_1(x, \lambda) &= k_{12} \cos(\lambda x; q) - k_{11} \sin(\lambda x; q) \\
&+ q \int_0^x \left\{ \sin(\lambda x; q) \cos(\lambda q t; q) - \cos(\lambda x; q) \sin(\lambda q t; q) \right\} p(q t) \phi_1(q t, \lambda) \, dq \, dt, \\
\phi_2(x, \lambda) &= -k_{12} \sqrt{q} \sin(\lambda \sqrt{q} x; q) - k_{11} \cos(\lambda \sqrt{q} x; q) \\
&+ \sqrt{q} \int_0^x \left\{ \cos(\lambda \sqrt{q} x; q) \cos(\lambda q t; q) + \sqrt{q} \sin(\lambda \sqrt{q} x; q) \sin(\lambda q t; q) \right\} p(q t) \phi_1(q t, \lambda) \, dq \, dt. 
\end{aligned}
\]  

(24) (25)

**Proof:** Let us construct two solutions of the \( q \)-system (1) as

\[
\begin{align*}
\phi_1(x, \lambda) &= \begin{pmatrix} \phi_{11}(x, \lambda) \\ \phi_{12}(x, \lambda) \end{pmatrix} = \begin{pmatrix} \cos(\lambda x; q) \\ -q \sin(\lambda \sqrt{q} x; q) \end{pmatrix}, \\
\phi_2(x, \lambda) &= \begin{pmatrix} \phi_{21}(x, \lambda) \\ \phi_{22}(x, \lambda) \end{pmatrix} = \begin{pmatrix} \sin(\lambda x; q) \\ \cos(\lambda \sqrt{q} x; q) \end{pmatrix},
\end{align*}
\]  

(26)

for \( p(x) = r(x) \equiv 0 \), with the \( q \)-Wronskian

\[ W(\phi_1, \phi_2)(x, \lambda) = \phi_{11}(x, \lambda) \phi_{22}(x q^{-1}, \lambda) - \phi_{21}(x, \lambda) \phi_{12}(x q^{-1}, \lambda) = 1. \]  

(27)

The function
\[ y(x, \lambda) = \begin{pmatrix} y_1(x, \lambda) \\ y_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} c_1 \cos(\lambda x; q) + c_2 \sin(\lambda x; q) \\ -c_1 \sqrt{q} \sin(\lambda \sqrt{q} x; q) + c_2 \cos(\lambda \sqrt{q} x; q) \end{pmatrix}, \tag{28} \]

is a fundamental set of the \( q \)-system (1) for \( p(x) = r(x) \equiv 0 \). Using \( q \)-analogue of the method of variation of constants, a particular solution of the \( q \)-system (1) may be given by

\[ \begin{align*}
 y_1(x, \lambda) &= c_1(x) \cos(\lambda x; q) + c_2(x) \sin(\lambda x; q), \\
 y_2(x, \lambda) &= -c_1(x) \sqrt{q} \sin(\lambda \sqrt{q} x; q) + c_2(x) \cos(\lambda \sqrt{q} x; q). \tag{29} \end{align*} \]

Hence the functions \( c_i(x)(i = 1, 2) \) satisfy the \( q \)-linear system of equations

\[ \begin{align*}
 \cos(\lambda x; q) D_q c_1(x) + \sin(\lambda x; q) D_q c_2(x) &= -r(x^{-1}) y_2(x^{-1}, \lambda), \\
 \sqrt{q} \sin(\lambda q^{-1/2} x; q) D_q c_1(x) - \cos(\lambda q^{-1/2} x; q) D_q c_2(x) &= -q p(x) y_1(x, \lambda). \tag{30} \end{align*} \]

Since the equality (27) satisfies, then (30) has a unique solution which leads

\[ \begin{align*}
 D_q^{-1} c_1(x) &= -r(x^{-1}) \cos(\lambda q^{-1/2} x; q) y_2(x^{-1}, \lambda) - q p(x) \sin(\lambda x; q) y_1(x, \lambda), \\
 D_q^{-1} c_2(x) &= q p(x) \cos(\lambda x; q) y_1(x, \lambda) - r(x^{-1}) \sqrt{q} \sin(\lambda q^{-1/2} x; q) y_2(x^{-1}, \lambda). \tag{31} \end{align*} \]

Using the formula (7) and replacing \( x \) by \( x q \) in (31), then we obtain

\[ \begin{align*}
 c_1(x) &= c_1 - q \int_0^x p(q t) \sin(\lambda q t; q) y_1(q t, \lambda) d_q t - \int_0^x r(t) \cos(\lambda \sqrt{q} t; q) y_2(t, \lambda) d_q t, \\
 c_2(x) &= c_2 + q \int_0^x p(q t) \cos(\lambda q t; q) y_1(q t, \lambda) d_q t - \int_0^x r(t) \sqrt{q} \sin(\lambda \sqrt{q} t; q) y_2(t, \lambda) d_q t, \tag{32} \end{align*} \]

when \( c_i(x)(i = 1, 2) \) are \( q \)-regular at zero (here \( c_1 := c_1(0), c_2 := c_2(0) \)). That is the general solution \( y(x, \lambda) = \begin{pmatrix} y_1(x, \lambda) \\ y_2(x, \lambda) \end{pmatrix} \) of the \( q \)-system (1) is obtained to be

\[ \begin{align*}
 y_1(x, \lambda) &= c_1 \cos(\lambda x; q) + c_2 \sin(\lambda x; q) \\
 &+ q \int_0^x \left\{ \sin(\lambda x; q) \cos(\lambda q t; q) - \cos(\lambda x; q) \sin(\lambda q t; q) \right\} p(q t) y_1(q t, \lambda) d_q t \\
 &- \int_0^x \left\{ \cos(\lambda x; q) \cos(\lambda \sqrt{q} t; q) + \sqrt{q} \sin(\lambda x; q) \sin(\lambda \sqrt{q} t; q) \right\} r(t) y_2(t, \lambda) d_q t, \tag{33} \end{align*} \]
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\[ y_2(x, \lambda) = -c_1 \sqrt{q} \sin(\lambda \sqrt{q} x; \lambda) + c_2 \cos(\lambda \sqrt{q} x; \lambda) + \int_0^x \left[ \cos(\lambda \sqrt{q} x; \lambda) \cos(\lambda q t; \lambda) + \sqrt{q} \sin(\lambda \sqrt{q} x; \lambda) \sin(\lambda q t; \lambda) \right] p(q t) y_1(q t, \lambda) d_q t \]  
(34)

\[ + \sqrt{q} \int_0^x \left[ \sin(\lambda \sqrt{q} x; \lambda) \cos(\lambda q t; \lambda) - \cos(\lambda \sqrt{q} x; \lambda) \sin(\lambda q t; \lambda) \right] r(t) y_2(t, \lambda) d_q t. \]

It is easy to determine \( c_1, c_2 \) for which \( \phi(x, \lambda) \) satisfies the \( q \)-system (1) and the conditions (23), then we obtain (24) and (25).

**Theorem 3.2.** As \( |\lambda| \to \infty \), the function \( \phi(x, \lambda) \) has the following asymptotic relations

\[ \phi_1(x, \lambda) = k_{12} \cos(\lambda x; \lambda) - k_{11} \sin(\lambda x; \lambda) + O \left( |\lambda|^{-1} \exp \left( \frac{-\left( \log |\lambda| x (1-q) \right)^2}{\log q} \right) \right), \]

(35)

\[ \phi_2(x, \lambda) = -k_{12} \sqrt{q} \sin(\lambda \sqrt{q} x; \lambda) - k_{11} \cos(\lambda \sqrt{q} x; \lambda) + O \left( |\lambda|^{-1} \exp \left( \frac{-\left( \log |\lambda| q^{1/2} x (1-q) \right)^2}{\log q} \right) \right), \]

(36)

where for each \( x \in (0, a] \) the \( O \)-terms are uniform on \( \{xq^n : n \in \mathbb{N}\} \).

**Proof.** Similar to asymptotic relations for \( q \)-Sturm-Liouville problems in [4] and from Corollary (2.1), (35) and (36) can be obtained easily.

\[ \Delta(\lambda) = k_{21} \phi_1(a, \lambda) + k_{22} \phi_2(aq^{-1}, \lambda). \]

(37)

Then \( \frac{d\Delta(\lambda)}{d\lambda} = k_{21} \frac{\partial \phi_1(a, \lambda)}{\partial \lambda} + k_{22} \frac{\partial \phi_2(aq^{-1}, \lambda)}{\partial \lambda} \). Let \( \lambda_0 \) be a double eigenvalue, and \( \phi^0(x, \lambda_0) \) one of the corresponding eigenfunctions. Then the conditions \( \Delta(\lambda_0) = 0, \frac{d\Delta(\lambda_0)}{d\lambda} = 0 \) should be fulfilled simultaneously, i.e.,
\[
\begin{align*}
    k_2 \phi_0^0 (a, \lambda_0) + k_2 \phi_2^0 \left(aq^{-1}, \lambda_0\right) &= 0, \\
    k_2 \frac{\partial}{\partial \lambda} \phi_0^0 (a, \lambda_0) + k_2 \frac{\partial}{\partial \lambda} \phi_2^0 \left(aq^{-1}, \lambda_0\right) &= 0.
\end{align*}
\]  

(38)

Since \(k_2\) and \(k_2\) cannot vanish simultaneously, it follows from (38) that

\[
\phi_0^0 (a, \lambda_0) \frac{\partial \phi_0^0 \left(aq^{-1}, \lambda_0\right)}{\partial \lambda} - \phi_2^0 \left(aq^{-1}, \lambda_0\right) \frac{\partial \phi_0^0 (a, \lambda_0)}{\partial \lambda} = 0.
\]

(39)

Now, differentiating the \(q\)-system (1) with respect to \(\lambda\), we obtain

\[
\left\{-\frac{1}{q} D_q \left(\frac{\partial y_2}{\partial \lambda}\right) + \left\{p(x) - \lambda\right\} \frac{\partial y_1}{\partial \lambda} = y_1, \right. \\
\left. D_q \left(\frac{\partial y_1}{\partial \lambda}\right) + \{r(x) - \lambda\} \frac{\partial y_2}{\partial \lambda} = y_2. \right\}
\]

(40)

Multiplying the \(q\)-system (1) and (40) by \(\frac{\partial y_1}{\partial \lambda}, \frac{\partial y_2}{\partial \lambda}, -y_1\) and \(-y_2\), respectively, adding them together and integrating with respect to \(x\) from 0 and \(a\), we obtain

\[
\left\{y_1 (xq^{-1}, \lambda) \frac{\partial y_1}{\partial \lambda} - y_1 (x, \lambda) \frac{\partial y_1}{\partial \lambda}\right\}^a_0 = \int_0^a \left\{y_1^2 (x, \lambda) + y_2^2 (x, \lambda)\right\} d_q x.
\]

(41)

Putting \(\lambda = \lambda_0\), taking into account that \(\left.\frac{\partial \phi_0^0 (x, \lambda_0)}{\partial \lambda}\right|_{x=0} = \left.\frac{\partial \phi_0^0 (x, \lambda_0)}{\partial \lambda}\right|_{x=0} = 0\), and using the equality (39), we obtain the relation

\[
\int_0^a \left\{\left(\phi_0^0 (x, \lambda_0)\right)^2 + \left(\phi_2^0 (x, \lambda_0)\right)^2\right\} d_q x = 0.
\]

(42)

Hence \(\phi_0^0 (x, \lambda_0) = \phi_2^0 (x, \lambda_0) \equiv 0\), which is impossible. Consequently \(\lambda_0\) must be a simple eigenvalue.

\[\square\]

**Theorem 3.3.** As \(|\lambda| \to \infty\) the function \(\Delta(\lambda)\) has the following asymptotic relation

\[
\Delta(\lambda) = k_2 \left\{k_{12} \cos (\lambda a; q) - k_{11} \sin (\lambda a; q) + O \left(|\lambda|^{-1} \exp \left(-\log |\lambda| a (1-q)^2 \right)\right)\right\}
\]

\[+ k_2 \left\{-k_{12} \sqrt{q} \sin (\lambda q^{1/2} a; q) - k_{11} \cos (\lambda q^{1/2} a; q) + O \left(|\lambda|^{-1} \exp \left(-\log |\lambda| a q^{1/2} (1-q)^2 \right)\right)\right\}.
\]

(43)
Proof. The proof is immediate by substituting (35) and (36) into the relation
\[ \Delta(\lambda) = k_{21}\phi_1(a, \lambda) + k_{22}\phi_2(aq^{-1}, \lambda). \]

Theorem 3.4. The eigenvalues \( \{\lambda_m\} \) are the zeros of \( \Delta(\lambda) \) has the following asymptotic relations as \( m \to \infty \):

Case 1. \( k_{12} \neq 0, k_{11} = 0 \);

i) \( \lambda_m = \frac{q^{-m+1/2}}{a(1-q)} \left(1 + O(q^{m/2})\right), k_{21} = 0 \),

ii) \( \lambda_m = \frac{q^{-m+1/2}}{a(1-q)} \left(1 + O(q^{m})\right), k_{22} = 0 \).

Case 2. \( k_{12} = 0, k_{11} \neq 0 \);

i) \( \lambda_m = \frac{q^{-m+1}}{a(1-q)} \left(1 + O(q^{m/2})\right), k_{21} = 0 \),

ii) \( \lambda_m = \frac{q^{-m}}{a(1-q)} \left(1 + O(q^{m})\right), k_{22} = 0 \).

Proof. Similar to asymptotic relation for \( q \)-Sturm-Liouville problems [4] and from Theorem 2.1, the asymptotic relations (44)-(47) can be obtained easily.

Then from (35) and (36) and above theorem, the asymptotic relations of the eigenfunctions of the problem (1)-(3) is given by

Case 1. \( k_{12} \neq 0, k_{11} = 0 \);

\[
\phi(x, \lambda_m) = \begin{cases} 
\phi_1(x, \lambda_m) \\
\phi_2(x, \lambda_m)
\end{cases}
\]

\[
k_{12} \cos(\lambda_m x; q) + O(\lambda_m^{-1} \exp(\frac{-\log|x(1-q)|}{\log q})),
\]

\[
-k_{12} \sqrt{q} \sin(\lambda_m \sqrt{q} x; q) + O(\lambda_m^{-1} \exp(\frac{-\log|xq^{-1/2}x(1-q)|^2}{\log q})),
\]
Case 2. \( k_{12} = 0, k_{11} \neq 0; \)

\[
\phi(x, \lambda_m) = \begin{pmatrix}
\phi_1(x, \lambda_m) \\
\phi_2(x, \lambda_m)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-k_{11} \sin(\lambda_m x; q) + O\left( |\lambda_m|^{-1} \exp\left( -\frac{\log|\lambda_m| x (1-q)}{\log q} \right) \right), \\
-k_{11} \cos(\lambda_m \sqrt{q} x; q) + O\left( |\lambda_m|^{-1} \exp\left( -\frac{\log|\lambda_m| q^{1/2} x (1-q)}{\log q} \right) \right)
\end{pmatrix}.
\]

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